Asymptotically Optimal Load Balancing in Large-scale Heterogeneous Systems with Multiple Dispatchers

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IFIP Performance'20

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Joint work with...

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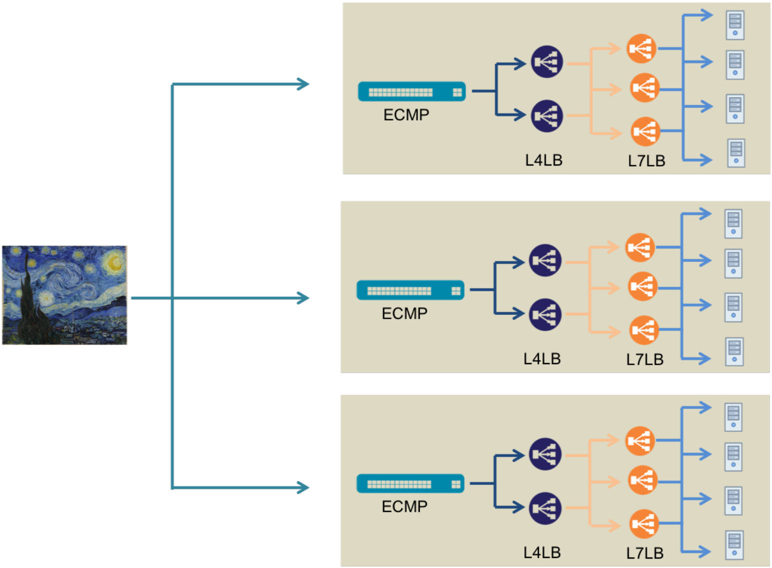
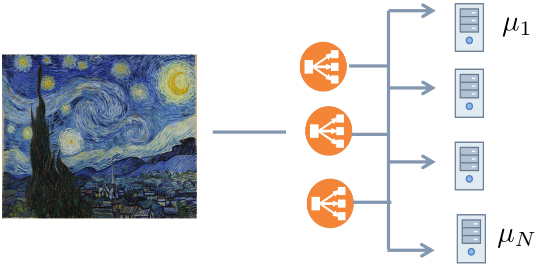
Load Balancing...



The Building Block...



Key features:

- Multiple dispatchers
- Heterogeneous servers

Motivating Questions...

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   - Stability? or even delay?

Our Proposed Design Framework: LED

The Local-Estimation-Driven (LED) framework...

1. **Memory:** Each dispatcher has a local memory storing its own estimates of each server's queue length (often outdated)
2. **Dispatching:** the dispatching decision at each dispatcher is made purely based on local memory
3. **Updating:** the local memory is updated with the true queue length via messages between dispatchers and servers

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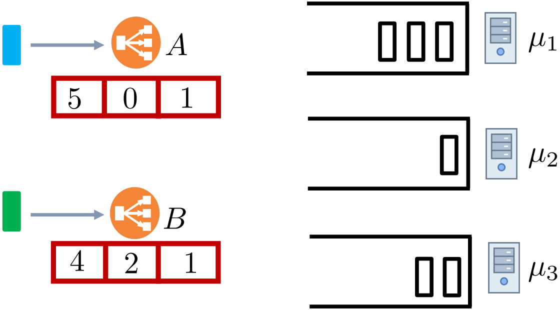
Key contributions...

1. Sufficient conditions on dispatching and updating strategies: throughput optimality and delay optimality in heavy traffic
2. Shed light on recently proposed open problem on LB with delayed information [David Lipshutz'19]

One Concrete Example...



Memory: Each dispatcher keeps its own local estimates (often outdated)...

- Dispatcher A 'believes' that: server 1 with queue length 5, server 2 with 0, and server 3 with 1
- Dispatcher B 'believes' that: server 1 with queue length 4, server 2 with 2, and server 3 with 1

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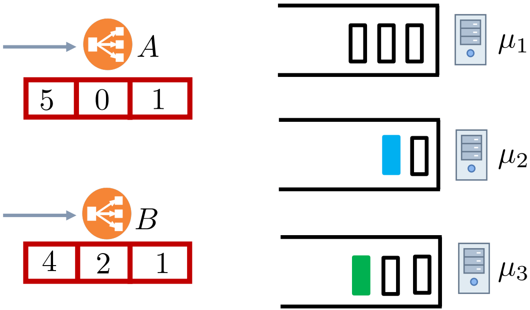
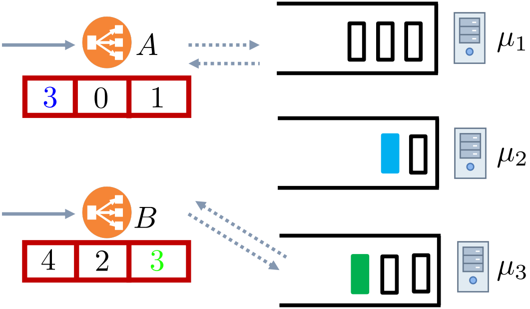


Dispatching strategy: Local-Join-Shortest-Queue (L-JSQ)

- each dispatcher independently routes new arrivals to the server with the shortest local estimates
- e.g., Dispatcher A routes to server 2, Dispatcher B routes to server 3

One Concrete Example...



Updating strategy: Push-based update via sampling

- each dispatcher independently randomly samples d servers with probability p
- update its corresponding local estimates with the true queue lengths

Related Works...

1. LB in multiple dispatchers:
   - JIQ in [Lu et al' 11]: consider homogeneous servers; JIQ is unstable in general for fixed number of heterogeneous servers [Zhou et al' 17]
   - Pull-based algorithm in [Stolyar' 17]: heterogeneous server pools in the large-system regime; assume homogeneous loads across dispatchers

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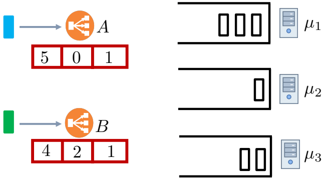
   - All of them consider a single dispatcher
3. Most related to ours is the recent work [Vargaftik et al' 20]
   - They only consider one particular dispatching strategy, i.e., Local-JSQ.
   - They only investigate stability

Model...



- M dispatchers and N servers in discrete-time.
- **Arrival:** total number of arriving tasks $A_\Sigma(t)$ with rate λ_Σ, general distribution [1]
  - $A_\Sigma(t)$ integer-valued *i.i,d* across time-slots
  - $A_\Sigma(t) = \sum_{m=1}^{M} A^m(t)$, $A^m(t)$ arrivals at dispatcher m
  - assume $\mathbb{P}\left(A^m(t) > 0\right) \geq p_0 > 0, \quad \forall (m,t) \in \mathcal{M} \times \mathbb{N}$,
- **Service:** average number of tasks can be served at server k is μ_k, general distribution.
  - $S_n(t)$ is integer-valued, *i.i.d* across time and independent of arrival and queue lengths
- **Memory:** $\widetilde{\mathbf{Q}}^m(t)) = (Q_1^m(t), \ldots, Q_N^m(t))$
- **System states:** $Z(t) = (\mathbf{Q}(t), \{\widetilde{\mathbf{Q}}^1(t))), \ldots, \widetilde{\mathbf{Q}}^M(t)\})$

[1] with all moments bounded by absolute constants

Metrics...

In this paper, we consider both throughput optimality and delay optimality in heavy traffic...

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Definition (Throughput Optimality)

A LB policy is throughput optimal if the system is positive recurrent under any $\epsilon > 0$ and all the moments of $\|\overline{\mathbf{Q}}^{(\epsilon)}\|$ are finite

Note: this definition is stronger than simple stability

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Definition (Heavy-traffic Delay Optimality)

A LB policy is said to be heavy-traffic delay optimal in steady-state if the steady-state queue length vector $\overline{\mathbf{Q}}^{(\epsilon)}$ satisfies

$$\lim_{\epsilon \downarrow 0} \epsilon \mathbb{E}\left[\sum_{n=1}^{N} \overline{Q}_n^{(\epsilon)}\right] = \lim_{\epsilon \downarrow 0} \epsilon \mathbb{E}\left[\bar{q}^{\epsilon}\right],$$

where $\mathbb{E}\left[\bar{q}^{\epsilon}\right]$ is the mean queue length in resource-pooling system.

Resource-pooling system: pool all the service into one super single server

Dispatching Preference...

- Fix a dispatcher m, let $\sigma_t(\cdot)$ be a permutation of $(1, 2, \dots, N)$ that satisfies

$$\widetilde{Q}^m_{\sigma_t(1)}(t) \leq \widetilde{Q}^m_{\sigma_t(2)}(t) \leq \ldots \leq \widetilde{Q}^m_{\sigma_t(N)}(t).$$

- $P^m_n(t)$: probability of routing to server n at dispatcher m in time-slot t (again, based on local estimates)
- $\Delta^m_n(t)$: preference of the n-th shortest local estimate at dispatcher m, given by

$$\boxed{\Delta^m_n(t) := P^m_{\sigma_t(n)}(t) - \frac{\mu_{\sigma_t(n)}}{\sum \mu_n}}$$

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  - $\Delta^m_n(t) > 0$ means that policy has stronger preference of n-th shortest local estimates compared to (weighted) random routing
  - Note that $\sum_{n=1}^N \Delta^m_n(t) = 0$
  - Key: how to allocate the zero-sum?

δ-tilted Sum Condition

$$\Delta_n^m(t) := P_{\sigma_t(n)}^m(t) - \frac{\mu_{\sigma_t(n)}}{\sum \mu_n}$$

Definition

Fix a dispatcher m, for all $1 \leq j \leq N-1$, $\sum_{n=1}^{j} \Delta_n^m(t) \geq \delta$ for some constant $\delta \geq 0$ at each time-slot t.

Intuitions: for any first k ($k < N$) shortest local estimates, it has at least δ total preference

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Examples: suppose all μ_n are equal and $\widetilde{\mathbf{Q}}^m(t)) = (5, 0, 1)$

- δ-tilted Sum Condition satisfied with all $\mathbf{P}^m(t)$ s.t. for some $\delta \ge 0$
- $P_2^m(t) \ge \delta + 1/3$, $P_2^m(t) + P_3^m(t) \ge \delta + 2/3$, and $\sum P_n^m(t) = 1$

Implications:

- this condition also generalizes previous definition in [Zhou et al' 17,18]
- as a result, it allows us to establish new results (e.g., L-Pod), discussed later

Main Results

We have the following sufficient condition (informal) for throughput optimality...
Define: $\mathcal{I}_n^m(t)$ indicates server n's true queue length is updated at dispatcher m

Theorem

Consider an LED policy if

- *dispatching strategy satisfies δ-tilted sum condition for some $\delta \geq 0$*
- *updating strategy satisfies that $\mathbb{E}\left[\mathcal{I}_n^m(t) \mid Z(t)\right] > p$ for any $Z(t), m, n$ and some $p > 0$*

Then, it is throughput optimal

Remark:

- This directly generalizes LSQ policy in [Vargaftik et al' 20] in terms of stability

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- *both δ and p are independent of ϵ*

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Remark:

- This directly implies a large class of LED policies are heavy-traffic delay optimal, including the specific one LSQ in [Vargaftik et al' 20]
- This also sheds light on heavy-traffic delay optimality in delayed queue length information, raised in [David Lipshutz'19]
- Moreover, the single dispatcher with accurate information is just a special case of ours

Examples of 'nice' dispatching strategies

1. L-JSQ: Local-Join-Shortest-Queue (i.e., the LSQ in [Vargaftik et al' 20])
   - choose $i^* \in \arg\min_n \{\widetilde{Q}_n^m\}$
   - $\Delta_1^m(t) = 1 - \mu_{\sigma_t(1)}/\mu_\Sigma > 0$ and all others are less than 0
   - It can be easily seen that δ-tilted sum condition is satisfied

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2. L-JBA: Local-Join-Below-Average
   - Let $\bar{Q}^m(t) = \frac{1}{N}\sum_n \widetilde{Q}_n^m(t)$ and $\mathcal{A} := \{n : \widetilde{Q}_n^m(t) \leq \bar{Q}^m(t)\}$
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3. L-Pod: Local-Power-of-d
   - randomly samples d servers, join the one with the shortest local estimates
   - it turns out that even with heterogeneous servers, L-Pod can still satisfy δ-tilted sum as long as the services rates meet a certain condition

More on L-Pod

Proposition

Suppose the service rate vector $\boldsymbol{\mu} \in \mathbb{R}_+^N$ *satisfies*

$$\frac{\sum_{n=1}^{j} \mu_{[n]}}{\mu_\Sigma} + \delta \leq 1 - \frac{\binom{N-j}{d}}{\binom{N}{d}} \qquad \forall 1 \leq j \leq N-1, \tag{1}$$

for some constant $\delta \geq 0$*, in which* $\mu_{[n]}$ *is the n-th largest service rate. Then, L-Pod satisfies the* δ*-tilted sum condition.*

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for some constant $\delta \geq 0$*, in which* $\mu_{[n]}$ *is the n-th largest service rate. Then, L-Pod satisfies the* δ*-tilted sum condition.*

Remark:

- For the single dispatcher with accurate queue length information (which is a special case of ours), [Hurtado-Lange and Maguluri' 20]) derived similar conditions
- If $d = 1$, the only possible $\boldsymbol{\mu}$ and δ are $\mu_n = \mu$ for all n and $\delta = 0$
- If $d = N$, then all $\boldsymbol{\mu} \in \mathbb{R}_+^N$ satisfies (1) with $\delta = \mu_{min}/\mu_\Sigma > 0$

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   - thus, $\mathbb{E}\left[\mathcal{I}_n^m(t) \mid Z(t)\right] \geq p > 0$ is satisfied with $p = \tilde{p}d/N$

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   - thus, $\mathbb{E}\left[\mathcal{I}_n^m(t) \mid Z(t)\right] \geq p > 0$ is satisfied with $p = \tilde{p}d/N$
2. Pull-based: each server takes the initiative to sample dispatchers
   - e.g., at the end of each time-slot, if server n finishes one or more tasks, it randomly samples one dispatcher
   - if $Q_n = 0$, it reports w.p. 1
   - if $Q_n > 0$, it reports w.p. $\tilde{p} > 0$
   - it has been verified in [Vargaftik et al' 20]), this satisfies $\mathbb{E}\left[\mathcal{I}_n^m(t) \mid Z(t)\right] \geq p > 0$ for arbitrarily small $\tilde{p} > 0$

Of course, there are many more...

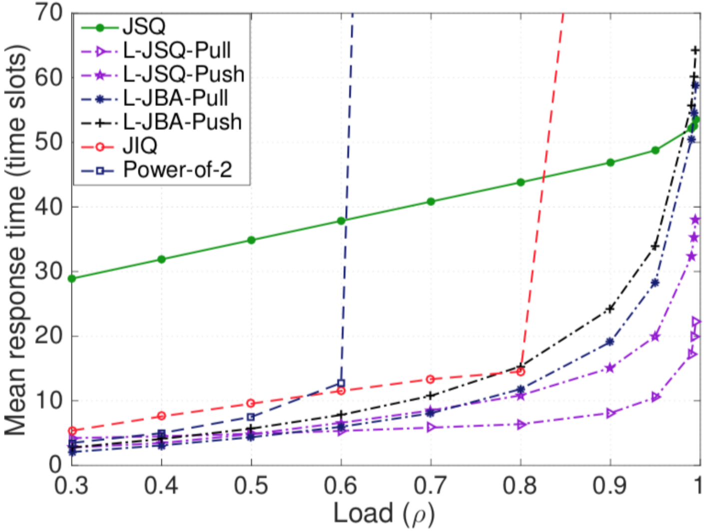
Recall our motivating questions

1. Question: With multiple dispatchers, does Join-Shortest-Queue still beat others in performance?
   - No, due to herd behavior, it actually behaves poorly
   - Thus, how can we avoid this?

   Answer: LED could be one solution due to its intrinsic randomness

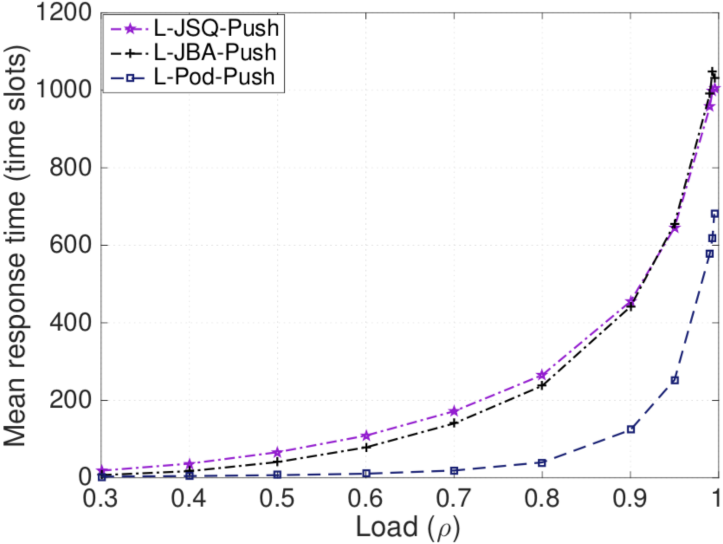
Inaccurate information helps...

100 heterogeneous servers, 10 dispatchers



Randomness further helps...

- 100 homogeneous servers, 10 dispatchers
- update probability is small $\tilde{p} = 0.01$



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   - Without communication across dispatchers

   Answer: For LED, we have
   - each dispatcher totally works independently
   - immediate dispatching, i.e., no waiting for update
   - simple and fast implementations, e.g., min-heap

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3. Question: How much communication between dispatchers and servers?
   - Minimize the messages between dispatchers and servers

   Answer: For LED, we have
   - the sampling and reporting probabilities can be arbitrarily small
   - of course, for practical performance, these parameters can be tuned to trade-off between messages and performance

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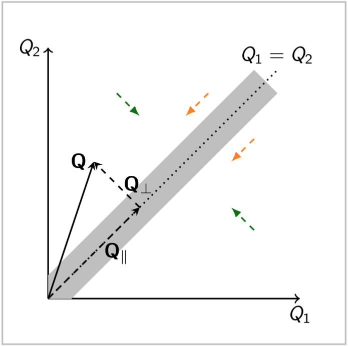
   - Stability? or even delay?

   Answer: For LED, we have
   - throughput optimality
   - delay optimality in heavy traffic

Main ideas behind proofs

- The main techniques are based on drift-based [Eryilmaz and Srikant'12])
- In particular, we utilize the sufficient conditions for throughput and heavy-traffic optimality in [Zhou et al'17], illustrated as follows



- Throughput optimality needs positive drift, obtained via

$$\sum_{t=1}^{T} \mathbb{E}\left[\langle \mathbf{Q}, \mathbf{A}-\mathbf{S}\rangle \mid \mathbf{Q}\right] \approx -\epsilon \|\mathbf{Q}\|$$

- Heavy-traffic optimality needs positive drift, obtained via

$$\sum_{t=1}^{T} \mathbb{E}\left[\langle \mathbf{Q}_\perp, \mathbf{A}-\mathbf{S}\rangle \mid \mathbf{Q}\right] \approx -\delta \|\mathbf{Q}_\perp\|$$

Main ideas behind proofs

Three additional challenges arise in our settings...

1. A more general dispatching condition (i.e., δ-tilted sum condition)
   - it exists even when the queue lengths are accurate
   - we draw inspirations from [Hurtado-Lange and Maguluri' 20]) to have a nice bound on the inner product between $\mathbf{Q}$ and $\mathbf{A}$
2. Outdated queue lengths information
   - our strategy is to do a decomposition
   - first, establish necessary drifts via dispatching strategy, assuming the queue lengths are accurate
   - second, bounding the error via update condition
3. System state includes local estimates
   - hence, for throughput optimality, they should also be bounded

Conclusion...

The LED combined with sufficient conditions give affirmative answers to all key questions...

1. Question: With multiple dispatchers, does Join-Shortest-Queue still beats others in performance?
   Answer: LED could be one solution due to its intrinsic randomness
2. Question: Can each dispatcher work independently with simple implementations?
   Answer: LED achieve independence, easy implementations
3. Question: How much communication between dispatchers and servers?
   Answer: LED, has the flexibility to tune the probability $\tilde{p}$
4. Question: Can we say something about performance guarantee?
   Answer: LED, can be throughput optimal and delay optimal in heavy traffic

Future Works

There are several interesting directions for LED...

1. Beyond the traditional heavy-traffic regime?
   - As pointed out by [Zhou et al' 18]), heavy-traffic delay optimal is a coarse metric in certain sense
   - How about waiting probability in large-system regimes?
2. How about continuous-time systems?
3. How about LED on graphs?
   - each node can serve a job or dispatches to neighbors
   - each node keeps local estimates of its neighbors
   - purely based on local memory to dispatch
   - infrequent update via communications between nodes

Thank you!

Q & A

Throughput optimality...

1. We consider the Lyapunov function
   $W(Z(t)) = \|\mathbf{Q}(t)\|^2 + \sum_{m=1}^{M} \left\| \mathbf{Q}(t) - \widetilde{\mathbf{Q}}^m(t) \right\|_1$
2. Let $X_n^m(t) \triangleq |Q_n(t) - \widetilde{Q}_n^m(t)|$, the drift is

$$D(Z(t_0)) = D_Q(t_0) + \sum_{m=1}^{M} \sum_{n=1}^{N} D_{X_n^m}(t_0) \tag{2}$$

   where

$$D_Q(t_0) \triangleq \mathbb{E}\left[\|\mathbf{Q}(t_0+T)\|^2 - \|\mathbf{Q}(t_0)\|^2 \mid Z(t_0) \right]$$
$$D_{X_n^m}(t_0) \triangleq \mathbb{E}\left[X_n^m(t_0+T) - X_n^m(t_0) \mid Z(t_0) \right]$$

3. $D_{X_n^m}(t_0) \leq -pX_n^m(t) + 2T\mu_\Sigma$

Throughput optimality (Cont'd)

4. Turn to $D_Q(t_0) \triangleq \mathbb{E}\left[\|\mathbf{Q}(t_0+T)\|^2 - \|\mathbf{Q}(t_0)\|^2 \mid Z(t_0)\right] \approx \sum \mathbb{E}\left[\langle \mathbf{Q}, \mathbf{A} - \mathbf{S} \rangle \mid Z(t_0)\right] + K$
5. We can decompose the first term into $(\beta_n^m(t) := P_n^m(t) - \mu_n/\mu_\Sigma)$

$$\text{RHS} \approx \underbrace{\sum_{t=t_0}^{t_0+T-1} \mathbb{E}\left[\sum_{n=1}^{N}\sum_{m=1}^{M}\left(Q_n(t) - \widetilde{Q}_n^m(t)\right)\beta_n^m(t)\lambda_m \mid Z\right]}_{\mathcal{T}_1}$$

$$+ \underbrace{\sum_{t=t_0}^{t_0+T-1} \mathbb{E}\left[\sum_{n=1}^{N}\sum_{m=1}^{M}\widetilde{Q}_n^m(t)\beta_n^m(t)\lambda_m \mid Z\right]}_{\mathcal{T}_2} - \frac{\epsilon\mu_{min}}{\mu_\Sigma}\|\mathbf{Q}(t_0)\|_1 .$$

6. For $\mathcal{T}_1$, by update condition, we have a constant bound on it

Throughput optimality (Cont'd)

7. Turn to $\mathcal{T}_2 = \sum_{t=t_0}^{t_0+T-1} \mathbb{E}\left[\sum_{n=1}^{N}\sum_{m=1}^{M} \widetilde{Q}_n^m(t)\beta_n^m(t)\lambda_m \mid Z\right]$
8. It is equal to $\sum_{t=t_0}^{t_0+T-1} \mathbb{E}\left[\sum_{n=1}^{N}\sum_{m=1}^{M} \widetilde{Q}_{\sigma_t(n)}^m(t)\Delta_n^m(t)\lambda_m \mid Z\right]$
9. The green term can be written as

$$\sum_{m=1}^{M}\left(\widetilde{Q}_{\sigma_t(1)}^m(t)\sum_{n=1}^{N}\Delta_n^m(t)\right) \tag{3}$$

$$+\sum_{m=1}^{M}\left(\sum_{k=2}^{N}\left(\sum_{n=k}^{N}\Delta_n^m(t)\right)\left(\widetilde{Q}_{\sigma_t(k)}^m(t)-\widetilde{Q}_{\sigma_t(k-1)}^m(t)\right)\right) \tag{4}$$

10. (3) is zero as $\sum_{n=1}^{N}\Delta_n^m(t) = 0$
11. (4) less than zero since $\sum_{n=k}^{N}\Delta_n^m(t) \le -\delta$ by δ-tilted sum condition

Heavy-traffic delay optimality...

1. We wish to establish $\sum_{t=t_0}^{t_0+T-1} \mathbb{E}\left[\langle \mathbf{Q}_\perp, \mathbf{A}-\mathbf{S}\rangle \mid Z\right] \approx -\delta' \|\mathbf{Q}_\perp\|$, δ' independent of ϵ
2. The key term $\sum_{t=t_0}^{t_0+T-1} \mathbb{E}\left[\langle \mathbf{Q}_\perp, \mathbf{A}\rangle \mid Z\right]$ can be written as

$$
\begin{aligned}
&\sum_{t=t_0}^{t_0+T-1} \mathbb{E}\left[\sum_{n=1}^{N} Q_{\perp,n}(t) \sum_{m=1}^{M} \beta_n^m(t)\lambda_m \mid Z\right] \\
=&\sum_{t=t_0}^{t_0+T-1} \mathbb{E}\left[\sum_{n=1}^{N}\sum_{m=1}^{M} \left(\widetilde{Q}_n^m(t) - \bar{Q}^m(t)\right) \beta_n^m(t)\lambda_m \mid Z\right] \quad (5)\\
&+\sum_{t=t_0}^{t_0+T-1} \mathbb{E}\left[\sum_{n=1}^{N}\sum_{m=1}^{M} \left(Q_n(t) - \widetilde{Q}_n^m(t)\right) \beta_n^m(t)\lambda_m \mid Z\right] \quad (6)\\
&+\sum_{t=t_0}^{t_0+T-1} \mathbb{E}\left[\sum_{n=1}^{N}\sum_{m=1}^{M} \left(\bar{Q}^m(t) - Q_{\text{avg}}(t)\right) \beta_n^m(t)\lambda_m \mid Z\right]. \quad (7)
\end{aligned}
$$

   where $\bar{Q}^m(t) := \frac{1}{N}\sum_n \widetilde{Q}_n^m(t)$ and $Q_{\text{avg}} := \frac{1}{N}\sum_n Q_n(t)$
3. By updating condition, (6) and (7) both can be upper bounded (properly chosen T)

Heavy-traffic delay optimality (Cont'd)

4. Turn to the green term, it can be written as

$$(5) = \sum_{t=t_0}^{t_0+T-1} \mathbb{E}\left[\sum_{m=1}^{M}\sum_{n=1}^{N} \widetilde{Q}^m_{\sigma_t(n)}(t)\Delta^m_n(t)\lambda_m \mid Z\right]$$

5. Follow the same decompositions as in (3) and (4), we have

$$(5) \le -\delta \sum_{t=t_0}^{t_0+T-1} \mathbb{E}\left[\sum_{m=1}^{M}\sum_{n=1}^{N} \widetilde{Q}^m_{max}(t) - \widetilde{Q}^m_{min}(t)\lambda_m \mid Z\right]$$

6. By a careful sample-path analysis, we have for some constant K

$$\begin{aligned}(5) &\le -\delta f(p)\lambda_{min}\left(Q^m_{max}(t_0) - Q_{min}(t_0)\right) + K \\ &\le -\delta' \left\|\mathbf{Q}_\perp(t_0)\right\| + K\end{aligned}$$