Distribution of Consensus in a Broadcasting-based Consensus-forming Algorithm

Shigeo Shioda

Graduate School of Science and Engineering, Chiba University

November 5th, 2020[1]

[1] IFIP WG 7.3 Performance 2020 conference (held online)

Background and Motivation

- In a consensus-forming algorithm [1], agents initially having different opinions mutually exchange and thereby update their opinions to achieve a consensus.
- In a broadcasting-based consensus-forming algorithm [2], at each discrete time, one agent is chosen randomly to broadcast its opinion to neighbors. Agents receiving an opinion compute the weighted average of their opinions and the received one.
- It is proved that the opinions of the agents almost surely converge to a consensus, but little is known about the statistical properties of the achieved consensus.

[1] M. H. Degroot, "Reaching a consensus," Journal of the American Statistical Association, vol. 69, issue. 345, 1974.
[2] T.C. Aysal, M.E. Yildiz, A.D. Sarwate,and A. Scaglione, "Broadcast gossip algorithms for consensus," IEEE Transactions on Signal processing vol. 57, no. 7, 2009.

Purpose

- To investigate the statistical properties of the consensus in a broadcasting-based consensus-forming algorithm.
- For this purpose,
  - fundamental equations (fixed-point equations) concerning the consensus achieved in the algorithm are derived.
  - based on the derived equations, two extreme cases (consensus forming by two agents and consensus forming by an infinite number of agents) are investigated.

Formulation of the Problem

- N agents interact over a directed graph with adjacency matrix $A = \{a_{ij}\}$.

- At each discrete time, one agent broadcasts its opinion. The probability that agent i broadcasts its opinion at a given time is denoted by $p_i > 0$.

- Opinions of agents at time $n(= 0, 1, 2, \dots)$ is denoted by $\boldsymbol{x}(n) \stackrel{\text{def}}{=} (x_0(n), \dots, x_{N-1}(n))$, where $x_i(n) \in \mathbb{R}$ is the opinion of agent i.

Formulation of the Problem

- Opinion updating at time n is expressed by

$$\boldsymbol{x}(n+1)^\top = Q^{(e_n)}\boldsymbol{x}(n)^\top,$$

where e_n denotes the agent broadcasting its opinion at time n and $Q^{(k)} = \{q_{ij}^{(k)}\}$ is the matrix with the following elements:

$$q_{ij}^{(k)} = \begin{cases} 1 - w_{ki}a_{ki}, & i = j \\ w_{ki}a_{ki}, & i \neq j, j = k \\ 0. & \text{otherwise} \end{cases}$$

- w_{ji} is a parameter indicating the degree to which agent j influences the opinion of agent i.

Additional Assumptions and Notation

- Assume a mesh and a homogeneous network between agents.

$$\forall i, j \ (i \neq j), \quad w_{ij} = w, \quad a_{ij} = 1.$$

- X denotes the opinion broadcasted at time 0. Its distribution is given by

$$P(X = x_i(0)) = p_i, \quad i = 1, \ldots, N.$$

- The consensus is denoted by X_∞.

Fundamental Equations for the Consensus

Theorem 4.1

X_∞ satisfies the following fixed-point equation:

$$X_\infty \overset{d}{=} (1-w)X_\infty + wX.$$

The first and second terms on the right-hand side are statistically independent.

Theorem 4.2

$$X_\infty \overset{d}{=} w\sum_{k=0}^{\infty}(1-w)^k X_k,$$

where $X_0, X_1, \ldots$ are independent random variables and are identically distributed with X.

Outline of the Proof of Theorem 4.1

Consensus X_∞ can be expressed as

$$X_\infty = \boldsymbol{\pi} \left(\prod_{n=0}^{\infty} Q^{(e_n)} \right) \boldsymbol{x}^{\mathsf{T}}, \quad \prod_{n=0}^{\infty} Q^{(e_n)} \stackrel{\text{def}}{=} \lim_{m\to\infty} Q^{(e_m)} \dots Q^{(e_0)}.$$

where $\boldsymbol{x} \stackrel{\text{def}}{=} \boldsymbol{x}(0)$ is the initial opinion and $\boldsymbol{\pi}$ is the eigenvector of $\sum_{k=1}^{N} p_k Q^{(k)}$. We see

$$Q^{(e_0)}\boldsymbol{x}^{\mathsf{T}} = (1-w)\boldsymbol{x}^{\mathsf{T}} + w x_{e_0} \mathbf{1}^{\mathsf{T}}, \quad \left(\prod_{n=1}^{\infty} Q^{(e_n)} \right) \mathbf{1}^{\mathsf{T}} = \mathbf{1}^{\mathsf{T}}.$$

Thus,

$$X_\infty = (1-w)\boldsymbol{\pi} \left(\prod_{n=1}^{\infty} Q^{(e_n)} \right) \boldsymbol{x}^{\mathsf{T}} + w x_{e_0}.$$

Here, $\boldsymbol{\pi} \left(\prod_{n=1}^{\infty} Q^{(e_n)} \right) \boldsymbol{x}^{\mathsf{T}}$ is the consensus reached when the opinions at time 1 are equal to $\boldsymbol{x}$, which is equal to X_∞ in distribution, while x_{e_0} is equal to X in distribution. These observations complete the proof.

Additional Assumption and Notation

- Initial opinion of agent 0 is 0 and initial opinion of agent 1 is 1.
- $p_0 = p_1 = 0.5$ (agents 0 and 1 broadcast their opinions with probability 0.5 at each discrete time.)
- Define the mapping $f_w : [0, 1) \to [0, 1)$,

$$y = \sum_{k=1}^{\infty} \frac{\epsilon_k(y)}{2^k} \longrightarrow f_w(y) = w \sum_{k=1}^{\infty} (1 - w)^{k-1} \epsilon_k(y).$$

Note that f_w is the inverse of the generalized Cantor function when $w > 0.5$.

Properties of $f_w(y)$

- When $w \neq 0.5$, $f_w(y)$ is discontinuous at $y \in \Lambda$ and continuous on $[0,1) \setminus \Lambda$, where

$$\Lambda \stackrel{\text{def}}{=} \{y_{n,k} : n = 1, 2, \ldots,\ k = 0, 1, \ldots, 2^{n-1} - 1\}, \quad y_{n,k} \stackrel{\text{def}}{=} \frac{2k+1}{2^n}.$$

  When $w = 0.5$, $f_w(y) = y$ and thus $f_w(y)$ is continuous on $[0,1)$.

- When $w \geq 0.5$, $f_w(y)$ is strictly increasing.

- When $w < 0.5$, $f_w(y)$ jumps downward at discontinuous points.

Main Results

Theorem 5.3

$$P(X_\infty \leq x) = \mu_l \left(f_w^{-1} \left([0, x] \right) \right),$$

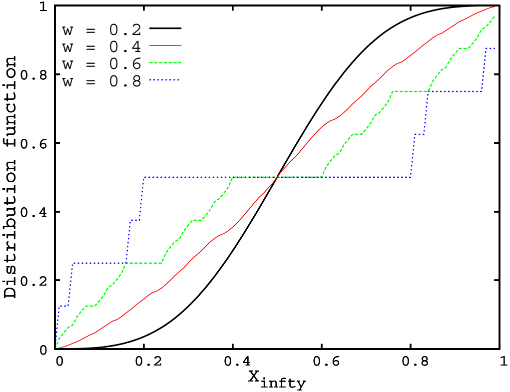
where μ_l denotes the Lebesgue measure.

Theorem 5.4

- If $w \leq 0.5$, then $X_\infty(\Omega) = [0, 1)$ and any $z \in [1, 0)$ could be an outcome of X_∞.
- If $w > 0.5$, then $\mu_l \left(X_\infty(\Omega) \right) = 0$ and the Lebesgue measure of the set of the possible outcomes of X_∞ is zero.

Numerical Examples

- Shape of the distribution function of X_∞ depends greatly on w.



Assumptions and Definitions

- Assume that X (opinion broadcasted at time 0) is a continuous random variable.
- This assumption corresponds to the case of letting the number of agents go to infinity.

Definition 6.1

Two random variables Z_1 and Z_2 are said to be of the same type if there exist constants $a > 0$ and $b \in \mathbb{R}$ with $Z_2 \overset{d}{=} aZ_1 + b$.

Definition 6.2

A random variable Z is stable if, for any positive constants a and b, $aZ_1 + bZ_2$ is the same type as Z, where Z_1 and Z_2 are independent copies of Z.

Consensus of an Infinite Number of Agents

Theorem 6.3

If X is stable, the consensus X_∞ is of the same type as X.

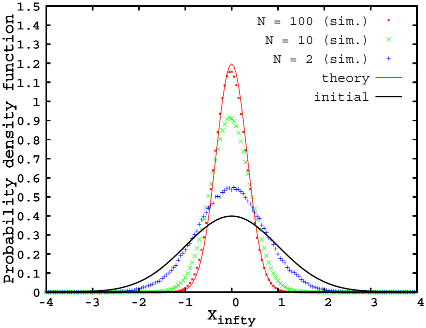
- Outline of the proof.
  - If X is stable, $w\sum_{k=0}^{\infty}(1-w)^k X_k$ is of the same type as X.
  - Thus, applying Theorem 4.2 completes the proof.
- A closed-form expression of the distribution of X_∞ can be derived if X follows a Gaussian, Cauchy, or Lévy distribution.

Case 1: Initial Opinions Follow a Gaussian Dist.

- If X follows a Gaussian distribution with mean μ and variance σ^2, the consensus also follows a Gaussian distribution with mean μ and variance $\sigma_\infty^2 = \frac{w\sigma^2}{2-w}$, where

$$\lim_{w \to 0} \sigma_\infty^2 = 0, \quad \lim_{w \to 1} \sigma_\infty^2 = \sigma^2.$$

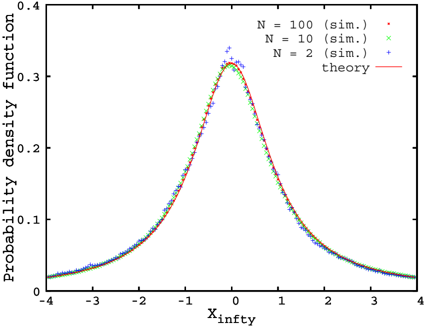
$w = 0.2$



Case 2: Initial Opinions Follow a Cauchy Dist.

- If X follows a Cauchy distribution, then the consensus is identically distributed with the initial opinion.

$w = 0.2$

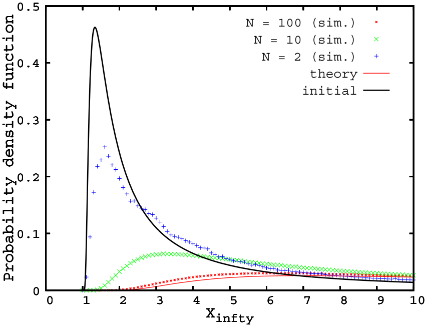


Case 3: Initial Opinions Follow a Lévy Dist.

- If X follows a Lévy distribution with location and scale parameters δ and γ, the consensus also follows a Lévy distribution with location and scale parameters δ and $\gamma_\infty \stackrel{\text{def}}{=} \frac{\gamma w}{2-w-2\sqrt{1-w}}$, where

$$\lim_{w\to 0} \gamma_\infty = \infty, \quad \lim_{w\to 1} \gamma_\infty = \gamma.$$

$w = 0.2$



Conclusion

- The statistical properties of the consensus in a broadcasting-based consensus-forming algorithm were investigated.
- In the infinite-number-of-agents case, if the initial opinions follow a stable distribution, then the consensus also follows a stable distribution.
- A closed-form expression for the distribution of the consensus exists if the initial opinions follow a Gaussian, Cauchy, or Lévy distribution.

Thank you very much.