Distribution of Consensus in a Broadcasting-based Consensus-forming Algorithm

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ABSTRACT

The consensus achieved in the consensus-forming algorithm is not generally a constant but rather a random variable, even if the initial opinions are the same. In the present paper, we investigate the statistical properties of the consensus in a broadcasting-based consensus-forming algorithm. We focus on two extreme cases: consensus forming by two agents and consensus forming by an infinite number of agents. In the two-agent case, we derive several properties of the distribution function of the consensus. In the infinite-number-of-agents case, we show that if the initial opinions follow a stable distribution, then the consensus also follows a stable distribution. In addition, we derive a closed-form expression of the probability density function of the consensus when the initial opinions follow a Gaussian distribution, a Cauchy distribution, or a Lévy distribution.

CCS Concepts

•**Mathematics of computing** → *Probabilistic algorithms;*

Keywords

broadcast, consensus, stable distribution, Gaussian, Cauchy, Lévy

1. INTRODUCTION

Consensus forming is a problem in which agents initially having different opinions mutually exchange and thereby update their opinions by a distributed algorithm to achieve a consensus [3]. Such problems have many applications, including distributed optimization, resource allocation in computer networks, distributed data fusion or clock synchronization in sensor networks, coordinate control of mobile agents, and opinion forming on social networks [2, 4, 1, 7, 6].

In the present paper, we consider the broadcasting-based consensus-forming algorithm. In this algorithm, one agent is chosen randomly with a given probability to broadcast its opinion to neighbor agents. Agents receiving an opinion from their common neighbor compute the weighted average of their opinions and the received opinion. The broadcasting-based algorithm has recently been investigated in several studies [2, 4, 7, 6] because the algorithm naturally arises in wireless networks, due to the broadcast nature of wireless communication [7]. The broadcasting-based algorithm also has high affinity to social networks because, for example, Twitter users very frequently broadcast a received tweet to all followers, which is usually referred to as retweeting.

Note that the consensus achieved in the randomized consensus forming algorithm, including the broadcasting-based consensus-forming algorithm, is not generally a constant, but rather a random variable. The achieved consensus depends on the order in which the agents broadcasts their opinions and it largely varies even if the initial opinions of agents are the same. However, little is known about the statistical properties of the achieved consensus.

In the present paper, we investigate the statistical properties of the consensus in the broadcasting-based consensus forming algorithm. We first derive fundamental equations concerning the consensus achieved in the algorithm. Based on the derived equations, we study the consensus for two extreme cases: consensus forming by two agents and consensus forming by an infinite number of agents. In the two-agent case, we obtain several results on the distribution function of the consensus. In the infinite-number-of-agents case, we show that if the initial opinions follow a stable distribution, the consensus also follows a stable distribution. We also derive a closed-form expression of the probability density function (PDF) of the consensus when the initial opinions follow a Gaussian distribution, a Cauchy distribution, or a Lévy distribution.

The remainder of the present paper is organized as follows. In Section 2, we present the problem formulation. In Section 3, we show some preliminary results for the convergence of the consensus. Then, in Section 4, we derive fundamental equations for the consensus. In Sections 5 and 6, we respectively show the results for the two-agent case and the infinite-number-of-agents case. Finally, in Section 7, we briefly conclude the paper.

2. PROBLEM FORMULATION

We consider N agents interacting over a directed graph. The agents are numbered from 1 to N, and the adjacency matrix of the directed graph connecting the agents is denoted by $A = \{a_{ij}\}$. If a directed link from agent $i \in \mathcal{N} \stackrel{\text{def}}{=} \{1, \dots, N\}$ to agent $j \in \mathcal{N}$ exists, $a_{ij} = 1$; otherwise, $a_{ij} = 0$. The agents have their own opinions, and the opinion of each agent is expressed as a real number. Let $x_i(n) \in \mathbb{R}$ denote the opinion of agent i at time $n \in \mathbb{N} \stackrel{\text{def}}{=} \{0, 1, 2, \dots\}$. The agents broadcast their opinions to each other as well as update their opinions at discrete time $n \in \mathbb{N}$ in the following manner. At each discrete time, one of the agents broadcasts

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its opinion to its neighbors. The probability that agent i broadcasts its opinion at each discrete time is denoted by p_i. We assume that $p_i > 0$ for all $i \in \mathcal{N}$ and $\sum_{i=1}^{N} p_i = 1$. At each discrete time n, agent i updates its opinion by the following equation:

$$x_i(n+1) = x_i(n) + \sum_{j=1}^{N} \delta_{j,e_n} a_{ji} w_{ji} (x_j(n) - x_i(n)), \quad (1)$$

where $e_n \in \mathcal{N}$ denotes the agent broadcasting its opinion at time n, and

$$\delta_{j,k} = \begin{cases} 1, & j = k \\ 0. & \text{otherwise} \end{cases}$$

Note that w_{ji} is a parameter indicating the degree to which agent j influences the opinion of agent i. We assume that $0 \le w_{ij} \le 1$ for all $i-j$ pairs. We also assume that $\{e_n\}_{n=0}^{\infty}$ are statistically independent random variables. Equation (1) can be expressed in matrix form as follows:

$$\boldsymbol{x}(n+1)^{\top} = Q^{(e_n)} \boldsymbol{x}(n)^{\top},$$
$$\boldsymbol{x}(n) \stackrel{\text{def}}{=} (x_0(n), \ldots, x_{N-1}(n)).$$

where $Q^{(k)}$ is a matrix expressing the opinion updates when agent k is broadcasting its opinion, and ij element of $Q^{(k)}$, $q_{ij}^{(k)}$, is given as

$$q_{ij}^{(k)} = \begin{cases} 1 - w_{ki} a_{ki}, & i = j \\ w_{ki} a_{ki}, & i \ne j, j = k \\ 0. & \text{otherwise} \end{cases}$$

Note that $Q^{(k)}$ is a stochastic matrix. The sum of the elements of each row vector of $Q^{(k)}$ is equal to 1. Define

$$Q \stackrel{\text{def}}{=} \sum_{k=1}^{N} p_k Q^{(k)}.$$

Here, Q is also a stochastic matrix. In the present paper, we assume that Q is irreducible and aperiodic, and without loss of generality, its elements are all positive[1]. We also assume that $x_i(0) < \infty$ for all $i \in \mathcal{N}$.

3. REACHING THE CONSENSUS

The broadcasting-based consensus-forming algorithm including the model described in Section 2 has been investigated in several papers [2, 4, 7, 6], and it has been proved that the opinions of the agents almost surely converge to a consensus. For the completeness of the present paper, in this section, we briefly show that the convergence of the opinions to a consensus is attained almost surely based on the model in Section 2. For this purpose, we define

$$M(n) \stackrel{\text{def}}{=} \max_{i \in \mathcal{N}} x_i(n), \quad m(n) \stackrel{\text{def}}{=} \min_{i \in \mathcal{N}} x_i(n),$$

and give two lemmas without proof.

Lemma 1. *$M(n+1) - m(n+1) \le M(n) - m(n)$ holds sample-path wise.*

[1]Because of the irreducibility, there exists $n \in \mathbb{N}$ such that the elements of Q^n are all positive, even if Q has a non-positive element. Thus, all results of the present paper hold by using Q^n instead of Q, even if Q has non-positive elements.

Lemma 2.

$$\lim_{n \to \infty} E[M(n) - m(n)] = 0.$$

The two lemmas readily yield the almost sure convergence to a consensus, as shown in the next theorem.

Theorem 1. The opinions of the agents converge to a consensus almost surely, i.e.,

$$\lim_{n \to \infty} (M(n) - m(n)) = 0, \quad w.p.1.$$

Proof. Lemma 1 means that $M(n) - m(n)$ is decreasing sample-path-wise. Thus, it follows from Lemma 2 and the monotone convergence theorem that

$$0 = \lim_{n \to \infty} E[M(n) - m(n)] = E[\lim_{n \to \infty} (M(n) - m(n))].$$

Since $M(n) - m(n) \ge 0$, it follows that

$$\lim_{n \to \infty} (M(n) - m(n)) = 0, \quad w.p.1.$$

$\square$

Let $\boldsymbol{\pi}$ be the left eigenvector of Q associated with eigenvalue 1. Assume that $\boldsymbol{\pi}$ is normalized such that the sum of elements is equal to 1, i.e., $\sum_{i=1}^{N} \pi_i = 1$. Letting $X(n) \stackrel{\text{def}}{=} \boldsymbol{\pi} \boldsymbol{x}(n)^{\mathsf{T}}$ yields

$$\begin{aligned} E[X(n)|X(m)] &= E[\pi Q^{n-m} \boldsymbol{x}(m)^{\mathsf{T}} | X(m)] \\ &= E[\pi \boldsymbol{x}(m) | X(m)] = X(m) \quad \text{for } m \le n. \end{aligned}$$

This means that $X(n)$ is a martingale. It follows from the martingale convergence theorem that $X(n)$ converges to a random variable X_∞ as $n \to \infty$ because $X(n) \le M(0) < \infty$ for all $n \in \mathbb{N}$. Since the opinions converge to a consensus (i.e., $x_1(\infty) = \cdots = x_N(\infty)$) with probability one, X_∞ is equal to the consensus.

4. FUNDAMENTAL EQUATIONS ON THE CONSENSUS

As shown in Section 3, the opinions of the agents converge to a consensus with probability one, and the consensus X_∞ is a random variable. In this section, we derive two fundamental equations to investigate the statistical properties of X_∞ under the assumption that

$$\forall i, j \ (i \ne j), \quad w_{ij} = w, \quad a_{ij} = 1.$$

Note that $w > 0$ due to the assumption of the irreducibility of Q. In the following, the initial opinions (opinions at time 0) of agents, $\boldsymbol{x}(0)$, are simply denoted as $\boldsymbol{x}$, and the consensus achieved starting from the initial opinions $\boldsymbol{x}$ is denoted by $X_\infty(\boldsymbol{x})$. Let X be a discrete random variable having the following distribution:

$$P(X = x_i) = p_i, \quad i = 1, \ldots, N,$$

where x_i is the initial opinion of agent i.

Theorem 2.

$$X_\infty(\boldsymbol{x}) \stackrel{d}{=} (1-w) X_\infty(\boldsymbol{x}) + wX, \quad (2)$$

where $\stackrel{d}{=}$ indicates equality in distribution. The first and second terms of the right-hand side of (2) are statistically independent.

PROOF. Consensus $X_\infty(\boldsymbol{x})$ can be expressed as

$$X_\infty(\boldsymbol{x}) = \boldsymbol{\pi} \left(\prod_{n=0}^{\infty} Q^{(e_n)} \right) \boldsymbol{x}^\mathsf{T},$$

$$\prod_{n=0}^{\infty} Q^{(e_n)} \stackrel{\text{def}}{=} \lim_{m\to\infty} Q^{(e_m)} \dots Q^{(e_1)} Q^{(e_0)}.$$

Observe that

$$Q^{(e_0)} \boldsymbol{x}^\mathsf{T} = (1-w)\boldsymbol{x}^\mathsf{T} + w x_{e_0} \mathbf{1}^\mathsf{T}, \tag{3}$$

and

$$\left(\prod_{n=1}^{\infty} Q^{(e_n)} \right) \mathbf{1}^\mathsf{T} = \mathbf{1}^\mathsf{T}, \tag{4}$$

where $\mathbf{1}$ is the row vector having all elements equal to 1. It follows from (3) and (4) that

$$X_\infty(\boldsymbol{x}) = (1-w)\boldsymbol{\pi} \left(\prod_{n=1}^{\infty} Q^{(e_n)} \right) \boldsymbol{x}^\mathsf{T} + w x_{e_0}. \tag{5}$$

Here, $\boldsymbol{\pi} \left(\prod_{n=1}^{\infty} Q^{(e_n)} \right) \boldsymbol{x}^\mathsf{T}$ is the consensus reached when the opinions at time 1 are equal to $\boldsymbol{x}$, which is equal to $X_\infty(\boldsymbol{x})$ in distribution, while x_{e_0} is equal to X in distribution. Note that the first term on the right-hand side of (5) depends on $(e_1, e_2, \dots)$, and the second term on the right-hand side of (5) depends only on e_0. Thus, the first and second terms are statistically independent. These observations complete the proof. $\square$

THEOREM 3.

$$X_\infty(\boldsymbol{x}) \stackrel{d}{=} w \sum_{k=0}^{\infty} (1-w)^k X_k, \tag{6}$$

where $X_0, X_1, \dots$ are independent random variables and are identically distributed with X.

PROOF. By substituting X into X_0 and $(1-w)X_\infty(\boldsymbol{x}) + wX_1$ into $X_\infty(\boldsymbol{x})$ in the right-hand-side of (2), we obtain

$$X_\infty(\boldsymbol{x}) \stackrel{d}{=} (1-w)^2 X_\infty(\boldsymbol{x}) + w(1-w)X_1 + wX_0.$$

By substituting $(1-w)X_\infty(\boldsymbol{x}) + wX_2$ into $X_\infty(\boldsymbol{x})$ in the right-hand-side of (2), we obtain

$$X_\infty(\boldsymbol{x}) \stackrel{d}{=} (1-w)^3 X_\infty(\boldsymbol{x}) + w(1-w)^2 X_2 + w(1-w)X_1 + wX_0.$$

Repeating the procedure mentioned above yields (6). $\square$

5. CONSENSUS BY TWO AGENTS

5.1 Expectation and Variance

We first focus on the simplest case, in which opinions are exchanged by two agents ($N = 2$). Without loss of generality, we assume that $\boldsymbol{x} = (0, 1)$ and $X_k : \Omega \to \{0, 1\}, k \in \mathbb{N}$ are random variables on probability space $(\Omega, \mathscr{F}, P)$. It readily follows from (6) that

$$E[X_\infty] = w \sum_{k=0}^{\infty} (1-w)^k E[X_k] = E[X],$$

$$Var[X_\infty] = w^2 \sum_{k=0}^{\infty} (1-w)^{2k} Var[X_k] = \frac{wVar[X]}{2-w}.$$

In other words, the expectation of the consensus does not depend on w, and the variance is an increasing function of w. In the remainder of the present paper, for simplicity, we assume that $P(X_k = 0) = P(X_k = 1) = 0.5$.

5.2 Distribution Function

In this subsection, we investigate the distribution of X_∞. For this purpose, we define the mapping $f_w : [0, 1) \to [0, 1)$

$$y = \sum_{k=1}^{\infty} \frac{\epsilon_k(y)}{2^k} \longrightarrow f_w(y) = w \sum_{k=1}^{\infty} (1-w)^{k-1} \epsilon_k(y)$$

Here, $\epsilon_k(y) \in \{0, 1\}$ is the digit associated with the term 2^{-k} in the binary representation of y. In order to make the binary representation unique, we do not allow an infinite string of 1's in the binary representation. For example, 0.5 has two binary representations:

$$0.5 = \frac{1}{2}, \quad 0.5 = \sum_{k=2}^{\infty} \frac{1}{2^k},$$

and we allow only the first representation. We define

$$\Lambda \stackrel{\text{def}}{=} \{y_{n,k} : n = 1, 2, \dots,\ k = 0, 1, \dots, 2^{n-1} - 1\},$$
$$y_{n,k} \stackrel{\text{def}}{=} \frac{2k+1}{2^n}.$$

Note that f_w is the inverse of the generalized Cantor function when $w > 0.5$. We show some properties of f_w as lemmas.

LEMMA 3. *When $w \neq 0.5$, $f_w(y)$ is discontinuous at $y \in \Lambda$ and continuous on $[0, 1) \setminus \Lambda$. When $w = 0.5$, $f_w(y) = y$, and thus $f_w(y)$ is continuous on $[0, 1)$.*

PROOF. It is easily seen that $f_w(y) = y$ when $w = 0.5$. Now, assume that $w \neq 0.5$. By expressing $k \in \{0, 1, \dots, 2^{n-1} - 1\}$ in a binary representation such that $k = \sum_{l=0}^{n-2} \xi_l(k) 2^l$, where $\xi_l(k) \in \{0, 1\}$, we have

$$y_{n,k} = \sum_{l=0}^{n-2} \frac{\xi_l(k) 2^{l+1}}{2^n} + \frac{1}{2^n} = \sum_{i=1}^{n-1} \frac{\xi_{n-i}(k)}{2^i} + \frac{1}{2^n}.$$

Let $y_{n,k}-$ be another (not allowed) binary representation of $y_{n,k}$.

$$y_{n,k}- \stackrel{\text{def}}{=} \sum_{i=1}^{n-1} \frac{\xi_{n-i}(k)}{2^i} + \sum_{i=n+1}^{\infty} \frac{1}{2^i}.$$

Observe that

$$f_w(y_{n,k}) = w \sum_{i=1}^{n-1} (1-w)^{i-1} \xi_{n-i}(k) + w(1-w)^{n-1},$$
$$\begin{aligned} f_w(y_{n,k}-) &= \lim_{\delta \downarrow 0} f_w(y_{n,k} - \delta) \\ &= w \sum_{i=1}^{n-1} (1-w)^{i-1} \xi_{n-i}(k) + w \sum_{i=n+1}^{\infty} (1-w)^{i-1} \\ &= w \sum_{i=1}^{n-1} (1-w)^{i-1} \xi_{n-i}(k) + (1-w)^n. \end{aligned}$$

and thus

$$f_w(y_{n,k}) - f_w(y_{n,k}-) = (1-w)^{n-1}(2w-1) \neq 0. \tag{7}$$

Since $y_{n,k} = y_{n,k}-$, $f_w(y)$ is discontinuous at $y_{n,k}$.

Next, we assume that $y \notin \Lambda$. For all $y \notin \Lambda$, there exists integer n satisfying the following two conditions:

1. $\epsilon_n(y) = 0$,
2. there is an integer $m > n$ satisfying $\epsilon_m(y) = 1$.

For n satisfying the above two conditions, we define

$$
\begin{aligned}
y_{n\uparrow}(y) &\stackrel{\text{def}}{=} \sum_{k=1}^{n} \frac{\epsilon_k(y)}{2^k} = \sum_{k=1}^{n-1} \frac{\epsilon_k(y)}{2^k}, \\
y_{n\downarrow}(y) &\stackrel{\text{def}}{=} \sum_{k=1}^{n} \frac{\epsilon_k(y)}{2^k} + \frac{1}{2^n} = \sum_{k=1}^{n-1} \frac{\epsilon_k(y)}{2^k} + \frac{1}{2^n} \\
&= y_{n\uparrow} + 2^{-n}, \\
y_{n\downarrow}(y)- &\stackrel{\text{def}}{=} \sum_{k=1}^{n-1} \frac{\epsilon_k(y)}{2^k} + \sum_{k=n+1}^{\infty} \frac{1}{2^k}.
\end{aligned}
$$

We see that $y \in (y_{n\uparrow}(y), y_{n\downarrow}(y))$ because of the conditions on n. We also see that

$$
\begin{aligned}
f_w(y_{n\uparrow}) &= w \sum_{k=1}^{n-1} (1-w)^{k-1} \epsilon_k(y) \\
&< f_w(y) < f_w(y_{n\downarrow}-) \\
&= \sum_{k=1}^{n-1} (1-w)^{k-1} \epsilon_k(y) + w \sum_{k=n+1}^{\infty} (1-w)^{k-1} \\
&= f_w(y_{n\uparrow}) + (1-w)^n.
\end{aligned}
$$

For any $\delta > 0$, we can choose $n > 0$ such that $(1-w)^n < \delta$, and the two conditions are satisfied. By choosing such n,

$$
z \in (y_{n\uparrow}(y), y_{n\downarrow}(y)) \Rightarrow |f_w(z) - f_w(y)| < (1-w)^n < \delta,
$$

which means that $f_w(y)$ is continuous on $[0, 1) \setminus \Lambda$. $\square$

REMARK 1. *When $w < 0.5$, $f_w(y_{n,k}) - f_w(y_{n,k}-) < 0$. In other words, $f_w(y)$ jumps downward at discontinuous points. When $w > 0.5$, $f_w(y)$ jumps upward at discontinuous points.*

LEMMA 4. *$f_w(y)$ is strictly increasing if $w \geq 0.5$.*

PROOF. It suffices to prove that $f_w(y_1) < f_w(y_2)$ for $y_1 < y_2$. If $y_1 < y_2$, then there exists $n \in \mathbb{N}$ such that

1. $\epsilon_k(y_1) = \epsilon_k(y_2)$ for $k = 1, 2, \cdots, n-1$, and
2. $\epsilon_n(y_1) = 0$, $\epsilon_n(y_2) = 1$

Therefore,

$$
\begin{aligned}
f_w(y_1) &= w \sum_{k=1}^{n-1} (1-w)^{k-1} \epsilon_k(y_1) + w \sum_{k=n+1}^{\infty} (1-w)^{k-1} \epsilon_k(y_1) \\
&< w \sum_{k=1}^{n-1} (1-w)^{k-1} \epsilon_k(y_1) + w \sum_{k=n+1}^{\infty} (1-w)^{k-1} \\
&= w \sum_{k=1}^{n-1} (1-w)^{k-1} \epsilon_k(y_1) + (1-w)^n \\
&\leq w \sum_{k=1}^{n-1} (1-w)^{k-1} \epsilon_k(y_1) + w(1-w)^{n-1} \\
&= w \sum_{k=1}^{n} (1-w)^{k-1} \epsilon_k(y_2) \leq f_w(y_2).
\end{aligned}
$$

where the second line follows from the fact that there exists $k > n$ such that $\epsilon_k(y_1) = 0$, and the fourth line follows from the fact that $1 - w \leq w$ for $w \geq 0.5$. $\square$

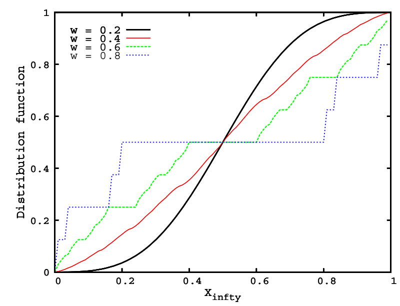

Figure 1: Distribution function of X_∞

In the next theorem, we define that $f_w^{-1}([0, x]) = \emptyset$ for $x < 0$ and $f_w^{-1}([0, x]) = [0, 1)$ for $x \geq 1$.

THEOREM 4.

$$
P(X_\infty \leq x) = \mu_l \left(f_w^{-1}([0, x]) \right), \tag{8}
$$

where μ_l denotes the Lebesgue measure.

PROOF. We define random variable Y on $(\Omega, \mathscr{F}, P)$ by

$$
Y(\omega) = \sum_{k=0}^{\infty} \frac{X_k(\omega)}{2^{k+1}}. \tag{9}
$$

Note that Y is uniformly distributed from 0 to 1, and thus $P(Y \in A) = \mu_l(A)$ for $A \subset [0, 1)$. We also note that the outcome of Y is given once, the outcome of $\{X_k\}_{k=0}^{\infty}$ is determined through (9), and thus X_∞ is also determined by (6). This relationship between Y and X_∞ can be represented by $f_w(Y) = X_\infty$. Thus, (8) follows from $\{X_\infty \leq x\} = \{Y \in f_w^{-1}([0, x])\}$. $\square$

We can obtain several results on the distribution of X_∞, as shown in the next theorem. We omit the proof of the theorem because of the space limitation.

THEOREM 5. If $w \leq 0.5$, then $X_\infty(\Omega) = [0, 1)$ and any $z \in [0, 1)$ could be an outcome of X_∞. If $w > 0.5$, then $\mu_l (X_\infty(\Omega)) = 0$ and the Lebesgue measure of the set of the possible outcomes of X_∞ is zero. In addition, the distribution of X_∞ is singular if $w > 0.5$.

When $w = 0.5$, X_∞ is uniformly distributed on $[0, 1]$. When $w = 2/3$, X_∞ follows Cantor distribution and the outcomes of X_∞ includes in Cantor set.

The distribution function of X_∞ can be approximately computed for any value of w with its error bound although the algorithm for the numerical computation is omitted because of the space limitation. Figure 1 shows the distribution of X_∞ for four different values of w (0.2, 0.4, 0.6, and 0.8) using the algorithm in the Appendix, where we set $n = 24$. As shown in the figure, the distribution function largely depends on w.

6. CONSENSUS BY AN INFINITE NUMBER OF AGENTS

The results obtained thus far are derived based on the model described in Section 2, where the number of agents is finite and X is a discrete random variable. In this section, we consider the cases in which X is now continuous with respect to the agents, which correspond to the limit cases where the number of agents tends toward infinity. In order to show the results, we introduce the notion of a stable distribution [5].

DEFINITION 1 ([5]). *Two random variables Z_1 and Z_2 are said to be of the same type if there exist constants $a > 0$ and $b \in \mathbb{R}$ with $Z_2 \overset{d}{=} aZ_1 + b$.*

DEFINITION 2 ([5]). *A random variable Z is stable if, for any positive constants a and b, $aZ_1 + bZ_2$ has the same type as Z where Z_1 and Z_2 are independent copies of Z.*

THEOREM 6. If X is stable, X_∞ is of the same type as X.

PROOF. If X is stable, $\tilde{X}_n \overset{\text{def}}{=} w\sum_{k=0}^{n}(1-w)^k X_k$ is of the same type as X. Since $\tilde{X}_n$ converges to X_∞ in distribution as $n \to \infty$, X_∞ is also of the same type as X. □

In the remainder of this section, we consider the distribution of the consensus X_∞ in the cases in which X (initial opinions of agents) follows a Gaussian distribution, a Cauchy distribution, or a Lévy distribution because, among stable distributions, these distributions have closed-form expressions.

6.1 Initial opinions follow a Gaussian distribution

Assuming that X follows a Gaussian distribution with average μ and variance σ^2, Theorem 6 readily yields that X_∞ also follows a Gaussian distribution. Let μ_∞ and σ_∞^2 be the average and variance, respectively, of X_∞. In order to calculate μ_∞ and σ_∞^2, observe that $w(1-w)^k X$ is a random variable following a Gaussian distribution with average $w(1-w)^k\mu$ and variance $w^2(1-w)^{2k}\sigma^2$. Thus, it follows from (6) that

$$\begin{aligned} \mu_\infty &= \mu\sum_{k=0}^{\infty} w(1-w)^k = \mu, \\ \sigma_\infty^2 &= \sigma_2\sum_{k=1}^{\infty} w^2(1-w)^{2k} = \frac{w\sigma^2}{2-w}, \end{aligned} \tag{10}$$

indicating that the average does not depend on w and the variance increases as w increases. In particular, when $w = 1$, the initial opinion and the consensus are identically distributed, and, in the limit of $w \to 0$, the consensus converges to a constant μ. This means that the variance of the consensus becomes smaller as the influence by the opinions of neighbors becomes smaller when the initial opinions follow a Gaussian distribution.

Figure 2 shows the results of simulation when the opinions of agents follow a Gaussian distribution with an average of zero and a variance of one. In the simulation, parameter w was set at 0.2, and N (number of agents) was set at three different values, 2, 10, or 100. At the beginning of the simulation, the opinion of each agent was independently given according to a Gaussian distribution, and the opinion exchange was conducted according to the procedure explained in Section 2 until the opinions of the agents converged to a consensus. This simulation was repeated 10^7 times by changing the initial opinions to obtain different consensus values and their distribution was computed. The simulation results were compared with the theoretical results, which is

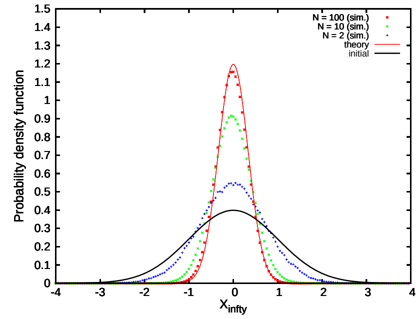

Figure 2: PDF of the consensus when the initial opinions follow a Gaussian distribution

a Gaussian distribution with the average and variance given by (10). In the figure, blue ($N = 2$), green ($N = 10$), and red ($N = 100$) dots denote the PDF of the consensus obtained by simulation, the red curve shows the PDF of the consensus by theory, and the blue curve shows the PDF of the initial opinion. As N becomes larger, the simulation results are more consistent with the theoretical results. When $N = 100$, the simulation results are almost identical to the theoretical results.

6.2 Initial opinions follow a Cauchy distribution

Next, we consider the case in which X follows a Cauchy distribution with location parameter δ and scale parameter γ: the PDF of a Cauchy distribution is given as

$$f_X(x) \overset{\text{def}}{=} \frac{1}{\pi}\frac{\gamma}{\gamma^2 + (x-\mu)^2}, \quad -\infty < x < \infty.$$

Theorem 6 yields that X_∞ also follows a Cauchy distribution. Let δ_∞ and γ_∞ be the location and scale parameter of X_∞. Note that $w(1-w)^k X$ follows a Cauchy distribution with location parameter $w(1-w)^k\delta$ and scale parameter $w(1-w)^k\gamma$, and that if $Z = X + Y$, where X and Y are mutually independent random variables following a Cauchy distribution, then the location parameter of Z is given by the sum of location parameters of X and Y and the scale parameter of Z is given by the sum of scale parameters of X and Y. Thus, it follows from (6) that

$$\delta_\infty = \delta\sum_{k=0}^{\infty} w(1-w)^k = \delta, \quad \gamma_\infty = \gamma\sum_{k=1}^{\infty} w(1-w)^k = \gamma.$$

That is, the consensus is identically distributed with the initial opinion regardless of the values of w.

Figure 3 shows the results of simulation when the opinion of the agents follows a Cauchy distribution with a location parameter of zero and a scale parameter of one. As with the simulation of Gaussian, parameter w was set at 0.2 and N was set at 2, 10, or 100. In the figure, the blue ($N = 2$), green ($N = 10$), and red ($N = 100$) dots denote the PDF of the consensus obtained by simulation, and the red curve shows the PDF of the consensus by theory (the initial opinion has the same PDF as the consensus in theory). The simulation results are well consistent with the theoretical results, regardless of N.

6.3 Initial opinions follow a Lévy distribution

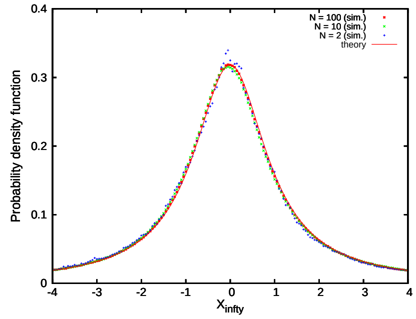

Figure 3: PDF of the consensus when the initial opinions follow a Cauchy distribution

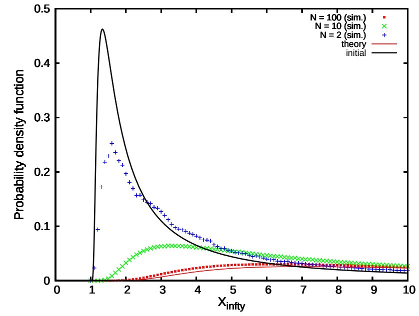

Figure 4: PDF of the consensus when the initial opinions follow a Lévy distribution

Finally, we consider the case in which X follows a Lévy distribution with location parameter δ and scale parameter γ: the PDF of the Lévy distribution is given as

$$f_X(x) \stackrel{\text{def}}{=} \sqrt{\frac{\gamma}{2\pi}} \frac{1}{(x-\mu)^{3/2}} e^{-\frac{\gamma}{2(x-\mu)}}, \quad \mu < x < \infty.$$

From Theorem 6, X_∞ follows a Lévy distribution. Note that $w(1-w)^k X$ follows a Lévy distribution with location parameter $w(1-w)^k\delta$ and scale parameter $w(1-w)^k\gamma$, and that if $Z = X + Y$, where X and Y are mutually independent random variables following a Lévy distribution, then the location parameter of Z is given by the sum of location parameters of X and Y and the square root of the scale parameter of Z is given by the sum of the square roots of the scale parameters of X and Y. Thus, it follows from (6) that

$$\delta_\infty = \delta \sum_{k=0}^{\infty} w(1-w)^k = \delta,$$

$$\gamma_\infty = \gamma \left(\sum_{k=0}^{\infty} \sqrt{w(1-w)^k} \right)^2 = \frac{\gamma w}{2 - w - 2\sqrt{1-w}}.$$

In particular,

$$\begin{aligned} \lim_{w\to 0} \gamma_\infty &= \lim_{w\to 0} \frac{\gamma w}{2 - w - 2\sqrt{1-w}} \\ &= \lim_{w\to 0} \frac{\gamma w}{\frac{w^2}{4} + o(w^3)} = \infty, \\ \lim_{w\to 1} \gamma_\infty &= \lim_{w\to 0} \frac{\gamma w}{2 - w - 2\sqrt{1-w}} = \gamma. \end{aligned}$$

In other words, when $w = 1$, the consensus is identically distributed with the initial opinions, and the scale parameter of the consensus increases as w decreases. In the limit of $w \to 0$, the scale parameter of the consensus tends toward infinity. This behavior is in sharp contrast to the case in which the distribution of the initial opinions is Gaussian.

Figure 4 shows the results of simulation when the opinions of the agents follow a Lévy distribution with a location parameter of one and a scale parameter of one. Parameter w was set at 0.2 and N was set at 2, 10, or 100. As N becomes larger, the simulation results are more consistent with the theory. When $N = 100$, the simulation results are almost identical to the theoretical results.

7. CONCLUDING REMARKS

In the present paper, we investigated the statistical properties of the consensus in the broadcasting-based consensus-forming algorithm. The assumption $P(X_k = 0) = P(X_k = 1) = 0.5$ made in the two-agent case could be relaxed by considering a de Rham distribution instead of the Lebesgue measure in Theorem 8. We also note that the two-agent case could be extended to multi-agent cases by imposing some assumption on the initial opinions, e.g., $x_i(0) = i$ for $i \in \mathcal{N}$. These possible extensions remain for future study.

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