Load balancing, redundancy, and multi type job and server systems

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Questions/Remarks welcome at : urtzi.ayesta@irit.fr

IFIP Performance 2020, 2/11/2020

Based on:

joint work with:

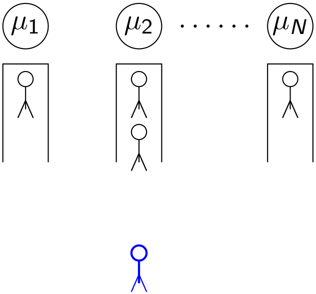
E. Anton, T. Bodas, J.L. Dorsman, M. Jonckheere, I.M. Verloop

and many other papers by:

Adan, Bonald, Busic, Comte, Gardner, Harchol-Balter, Hellemans, Hyytiä, Krzesinski, Mairesse, Moyal, Perry, Righter, Scheller Wolf, van Houdt, Visschers, Weiss

Animations by T. Bodas (Tikz !)

Load balancing



Very active research domain: JSQ, Power of d, Pull-Push based approaches, Jobs with multiple tasks, etc.

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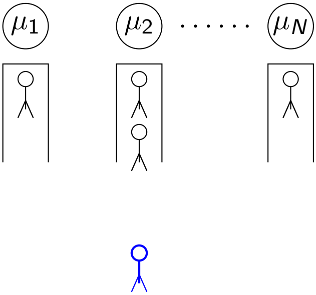

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$\Longrightarrow$ analysis often approximate or in limiting regimes

Product forms in load balancing systems

- Markovian queues: Queues that can be modeled as a *Markov chain*

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  → Poisson arrival rates

  → Exponential service requirements

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Product form

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Product form $\Rightarrow$ $\text{term}_1\ x\ \text{term}_2\ x \ldots\ \text{term}_n$

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Jackson's 1963 paper on product form queues was considered among Ten Most Influential Titles of Management Sciences First Fifty Years

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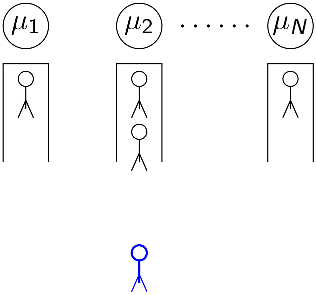
- Key Idea: Relate load balancing systems to a central queue architecture

Outline

- Redundancy
- Central Queue Architecture
- Order Independent Descriptor
  - Redundancy and cancel on complete
- Aggregated State Descriptor
  - Redundancy and cancel on start
- Generalizations:
  - Token-based framework
  - Generalized Order Independent
- Impact of assumptions: scheduling and independence

What is redundancy ?

Central idea: create several copies of the same job and use them to minimize latency !



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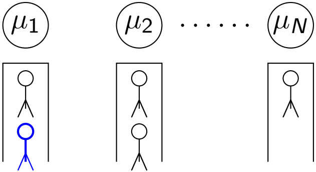

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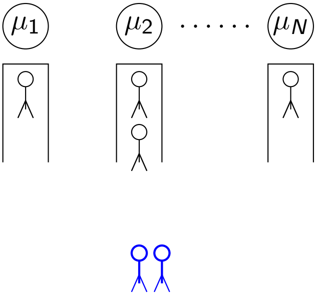

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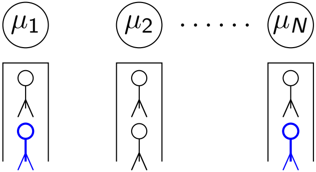

In practice deployed in DNS queries, search engines, Youtube, Mapreduce

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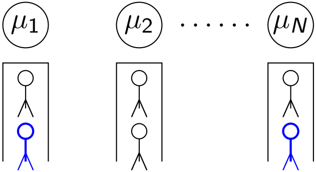
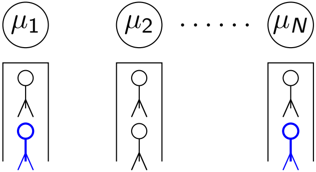


Exploit variability in the workload in different queues !

In practice deployed in DNS queries, search engines, Youtube, Mapreduce

When to cancel the redundant copy ?

A supermarket example



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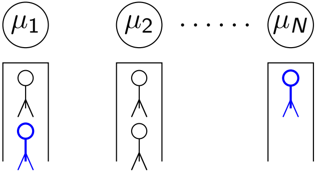

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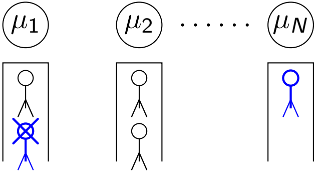

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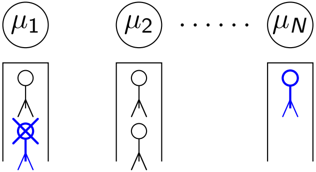
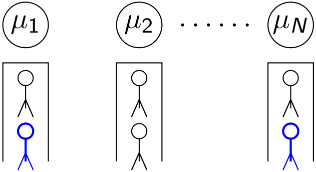


Cancel on start of service ($c.o.s.$ model)

When to cancel the redundant copy ?

The cancel-on-complete variant



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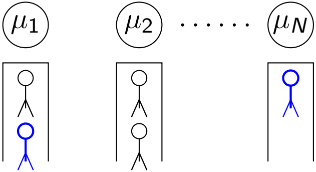

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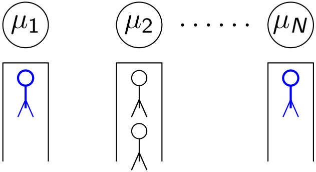

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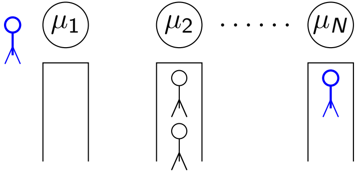

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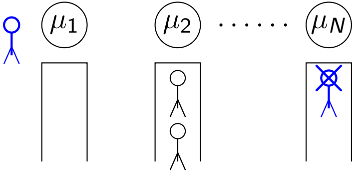

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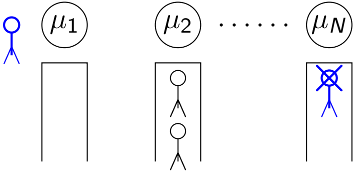
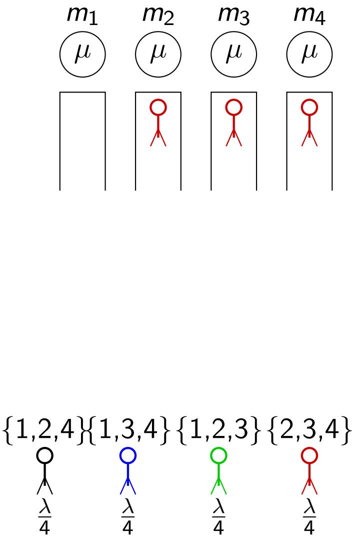


Cancel on completion of service (*c.o.c.* model)

c.o.c. and central queue



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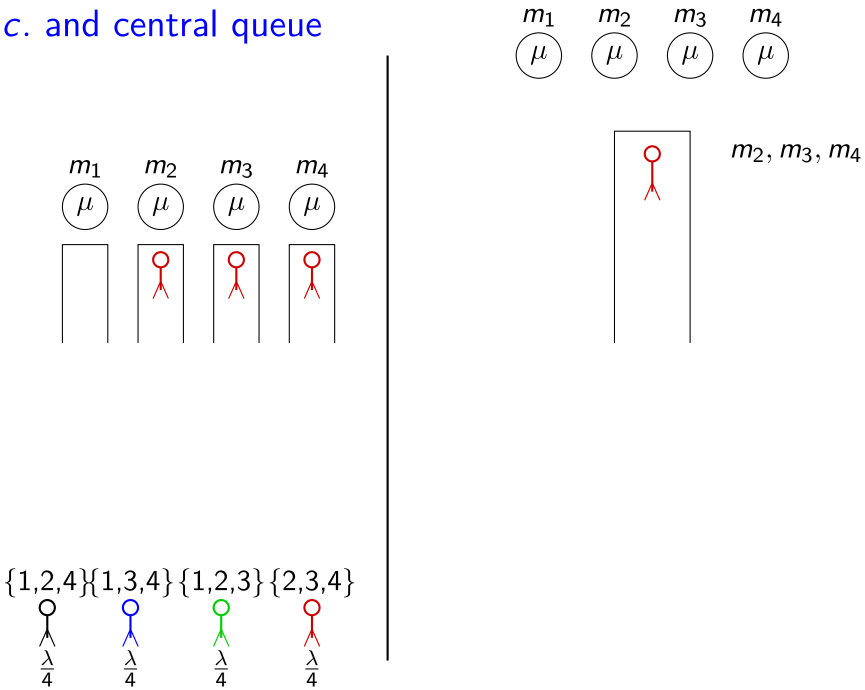

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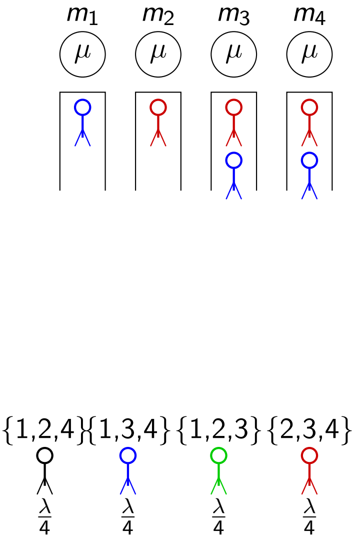

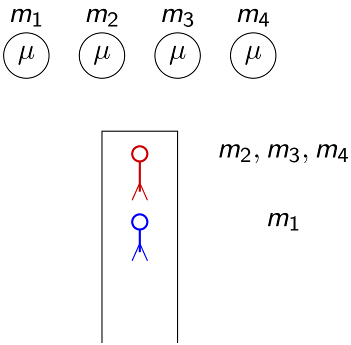

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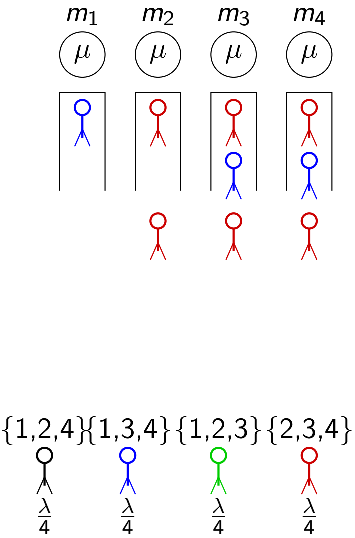

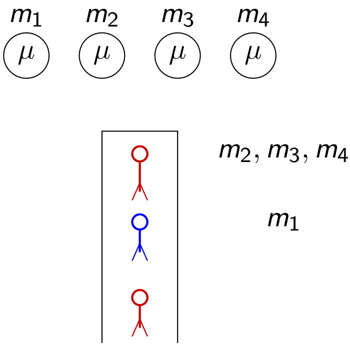

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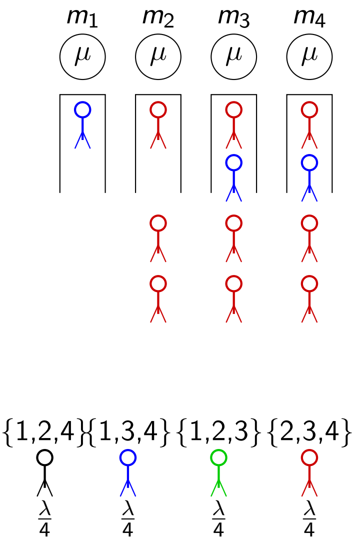

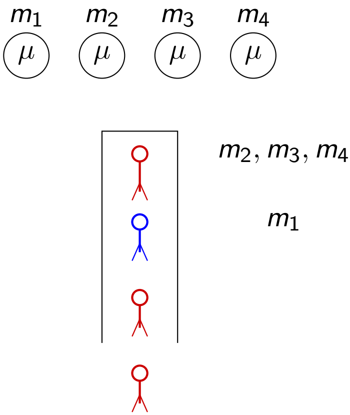

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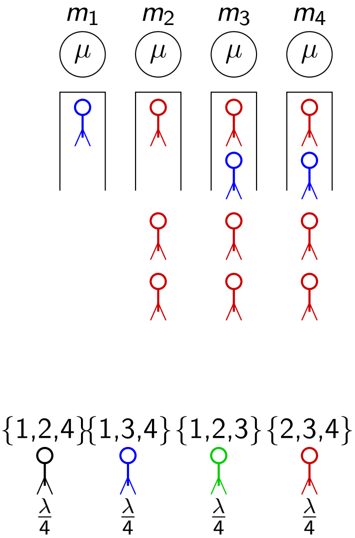
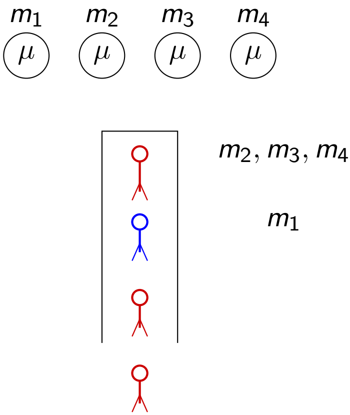
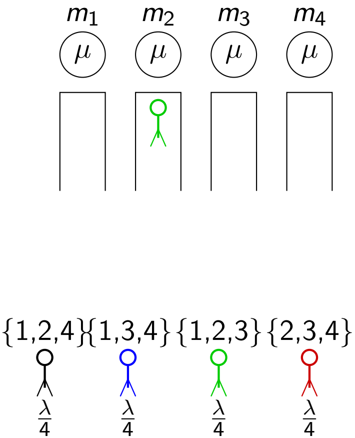




Gardner et al. QUESTA 2016

$c.o.s.$ and central queue

$N = 4$ and $d = 3$



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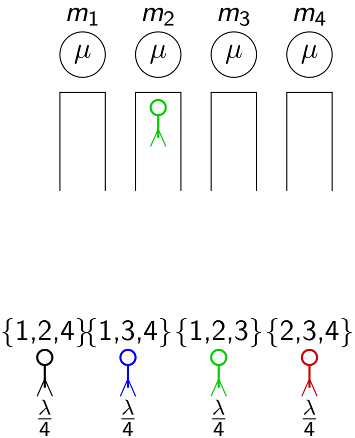

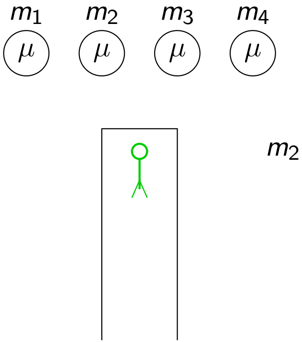

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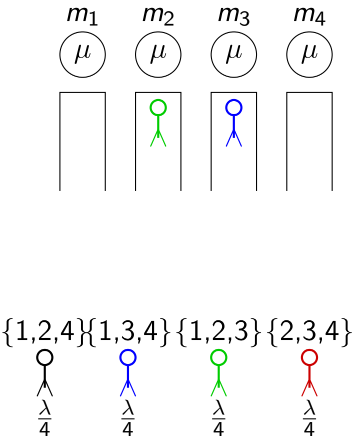

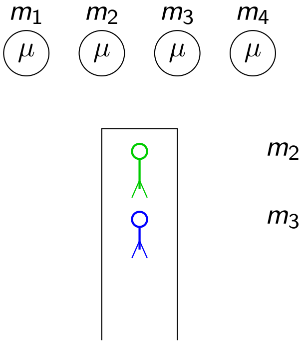

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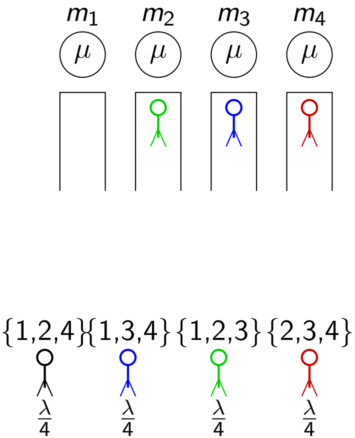

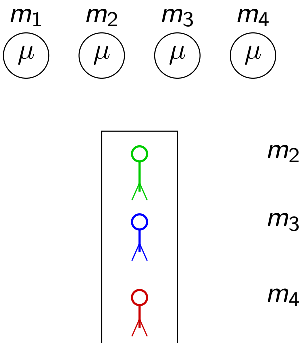

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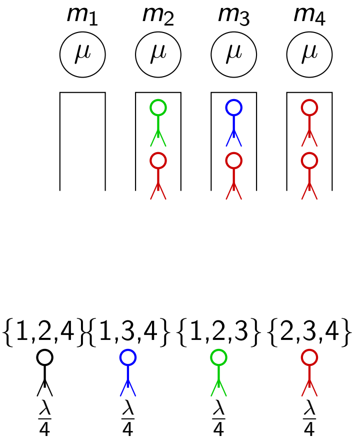

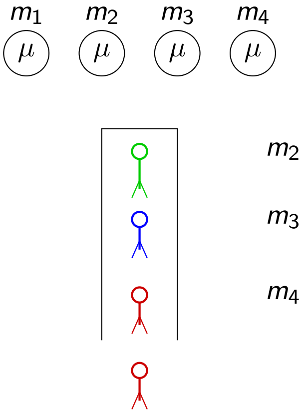

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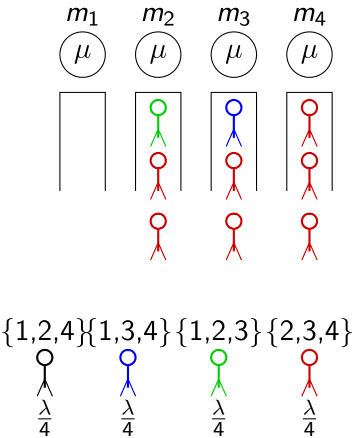

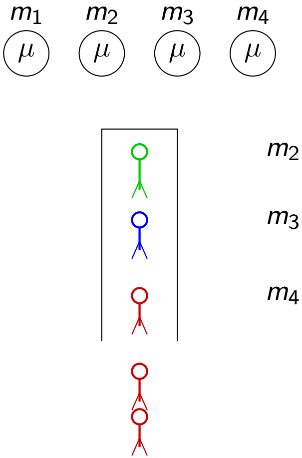

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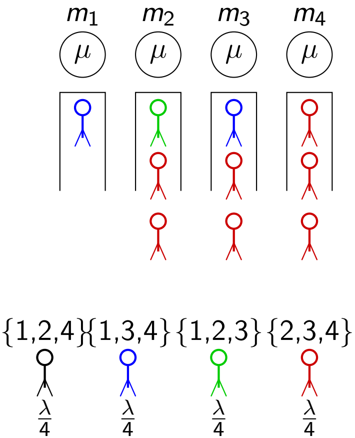

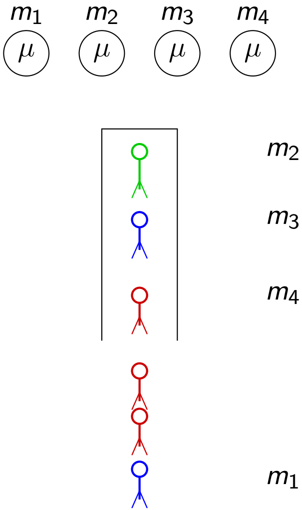

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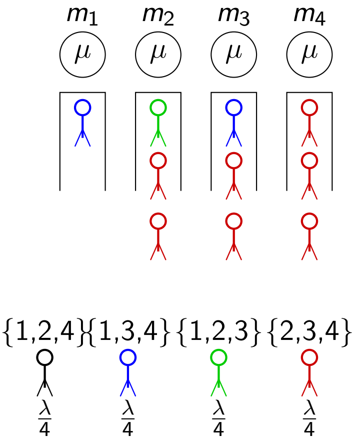

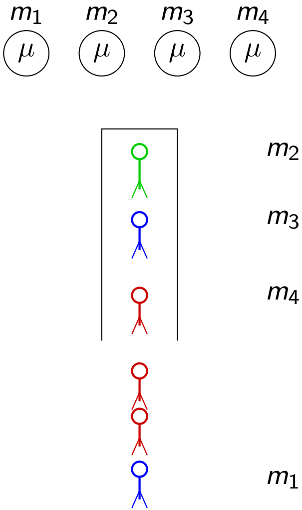

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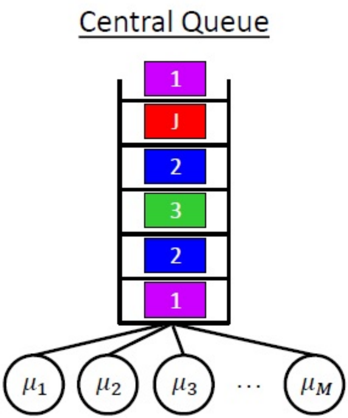
Central Queue



State descriptors:

- Job's point of view: $c(n) = (c_1, \ldots, c_n)$
- Server's point of view: $s = (n_1, M_1, \ldots, n_i, M_i)$

Order Independent Queues[1]

Classes $i = 1, \ldots, N$.
$\mathcal{S}_i$ set of servers that can process class-i
Service is FCFS

State descriptor: $c(n) = (c_1, \ldots, c_n)$.

$$\mu(c(n)) = \sum_{s \in \bigcup_{k=1}^{n} \mathcal{S}_{c_k}} \mu_s$$

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$$\mu(c_1, \ldots, c_k) - \mu(c_1, \ldots, c_{k-1}) = \sum_{s \in \mathcal{S}_{c_k} \setminus \bigcup_{l=1}^{k-1} \mathcal{S}_{c_l}} \mu_s$$

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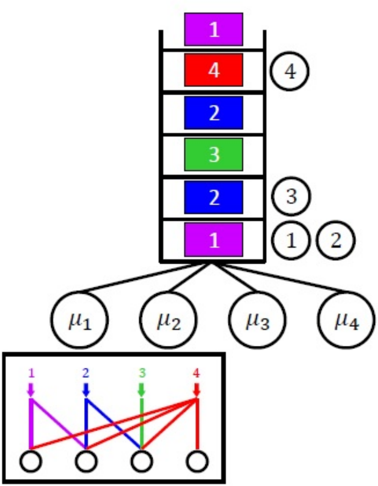
Order Independent Queues (example)[2]



State $c(6) = (1, 2, 3, 2, 4, 1)$

$\mu(c(6)) = \mu_1 + \mu_2 + \mu_3 + \mu_4$

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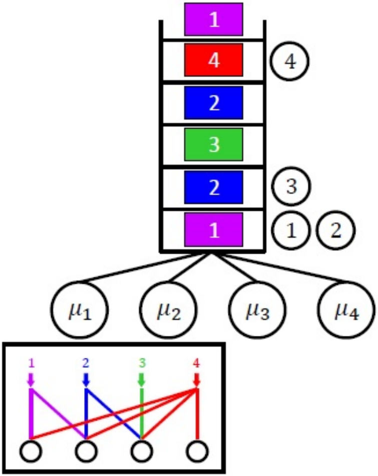

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$\mu(1, 2) - \mu(1) = \mu_3$

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OI queues

Key OI properties:

- For k-th job, its service rate $\mu(c_1, \ldots, c_k) - \mu(c_1, \ldots, c_{k-1})$ can only depend on what lies ahead.
- $\mu(c_1, \ldots, c_n) = \mu(c_{\sigma(1)}, \ldots, c_{\sigma(n)})$

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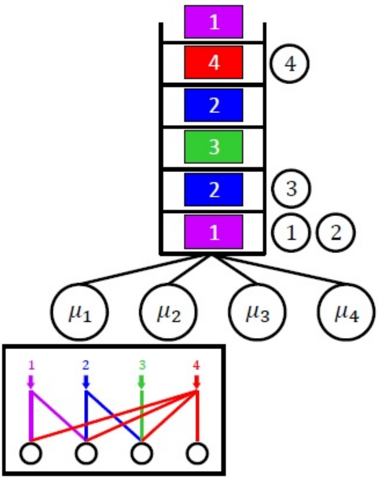
: Theorem: Given OI properties, the queue is quasi-reversible and the stationary distribution is:

$$\pi(c) = \pi(\emptyset) \prod_{k=1}^{n} \frac{\lambda_{c_k}}{\mu(c_1, \ldots, c_k)}$$

Order Independent Queues (example)



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$$\pi(c(6)) = \pi(\emptyset)\left(\frac{\lambda_1}{\mu_{1,2}}\right)\left(\frac{\lambda_2}{\mu_{1,2,3}}\right)\left(\frac{\lambda_3}{\mu_{1,2,3}}\right)\left(\frac{\lambda_2}{\mu_{1,2,3}}\right)\left(\frac{\lambda_4}{\mu}\right)\left(\frac{\lambda_1}{\mu}\right)$$

Sketch Proof of OI

Distribution satisfies partial balance equations:

rate out of $c(n)$ due to departure $=$ rate into $c(n)$ due to arrival

rate out of $c(n)$ due to class c arrival $=$ rate into $c(n)$ due to class c departure

[3] T. Bonald, C. Comte, F Mathieu, Performance of Balanced Fairness in Resource Pools: A Recursive Approach, Sigmetrics 2017

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Quasi reversible $\Longrightarrow$ Networks of queues have product form stationary distribution

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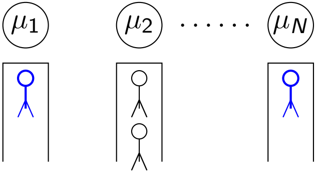
$\pi(\emptyset)$ can be computed recursively by removing a server s and all classes compatible with that server[3]

[3] T. Bonald, C. Comte, F Mathieu, Performance of Balanced Fairness in Resource Pools: A Recursive Approach, Sigmetrics 2017

Redundancy with *c.o.c.* is OI



$\mu(c_1, \dots, c_n) =$ sum of rates serving these classes

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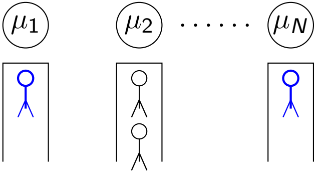
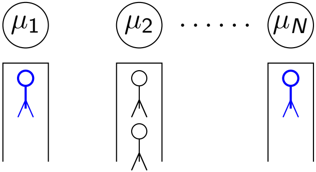


$\mu(c_1, \ldots, c_n)$ = sum of rates serving these classes

$\mu(c_1, \ldots, c_n) - \mu(c_1, \ldots, c_{n-1})$ = sum of rates of servers available to n class jobs

Redundancy with *c.o.c.* is OI



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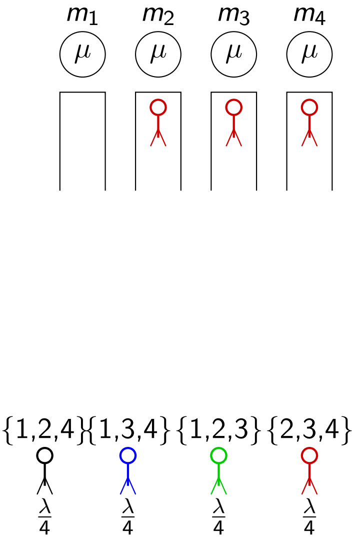
$\mu(c_1, \ldots, c_n) - \mu(c_1, \ldots, c_{n-1})$ = sum of rates of servers available to n class jobs

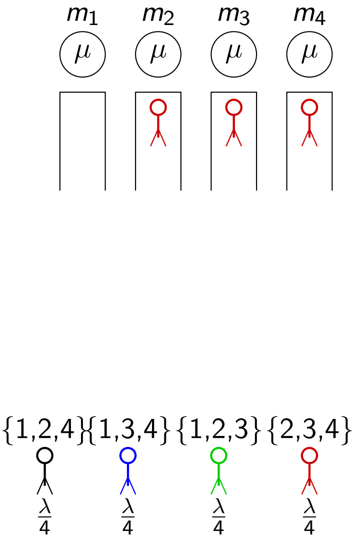
Steady-state distribution is product form!

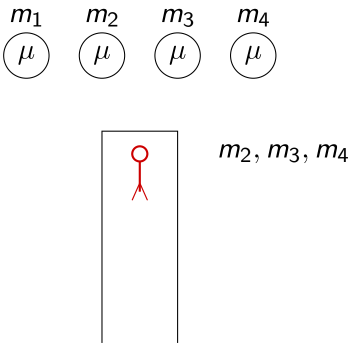
Key assumption: service time of copies independent!

c.o.c. and central queue

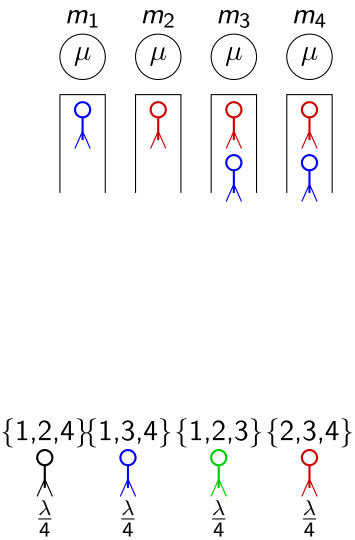


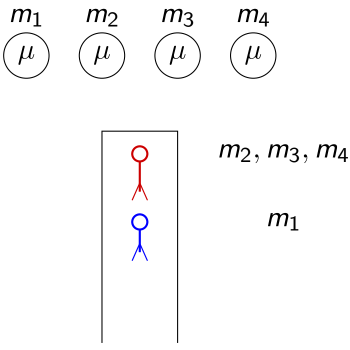
$c.o.c.$ and central queue





$c.o.c.$ and central queue





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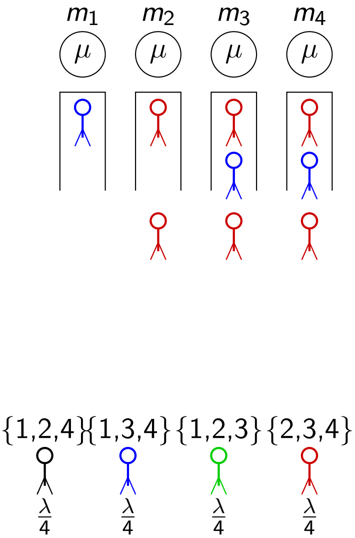
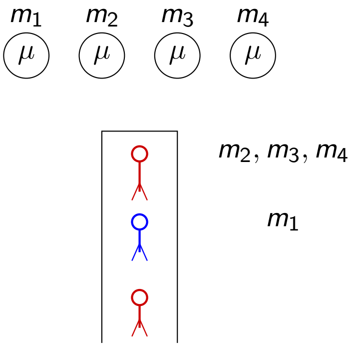
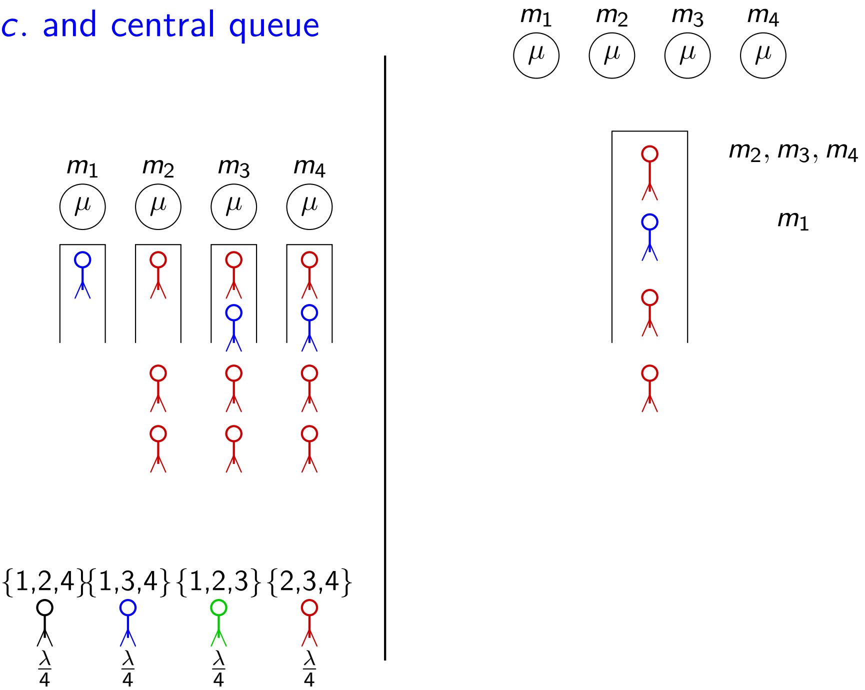
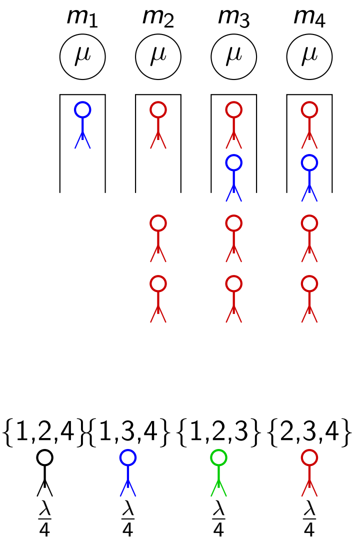


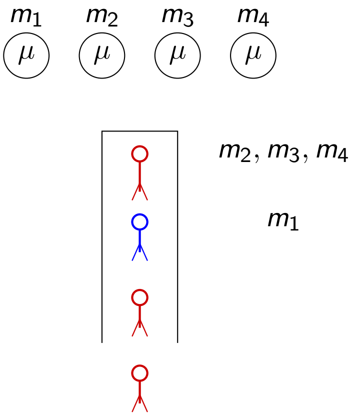


$c.o.c.$ and central queue



c.o.c. and central queue

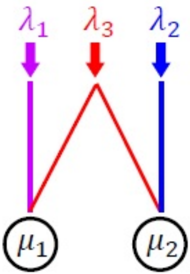




Gardner et al. QUESTA 2016

Performance Evaluation of Redundancy with *c.o.c.*

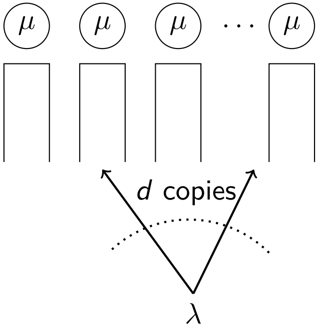
- Assume particular model: W, N etc.



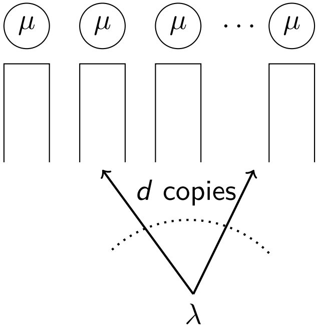
- Redundancy-d model[4]

[4] Gardner et al, Redundancy-d: The power of d choices for redundancy, OR 2017

redundancy-d with $c.o.c.$

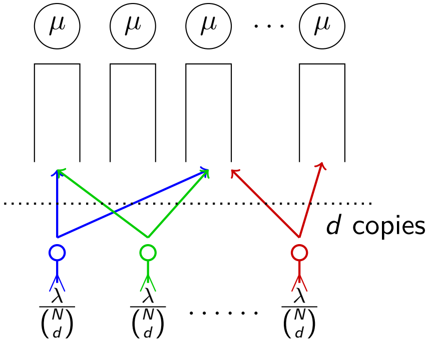


redundancy-d with $c.o.c.$



- N homogeneous servers with FIFO discipline
- Jobs arrivals are Poisson with rate λ
- Jobs have exponential service requirement
- Each arrival chooses d servers at random
- i.i.d. redundant copies for a job placed at d servers

redundancy-d with $c.o.c.$



- A special case of the generic multiclass model
- There are $\binom{N}{d}$ classes
- Each class has an arrival rate of $\frac{\lambda}{\binom{N}{d}}$

Performance Measures

Mean Delay

$$\mathbb{E}(T^{coc}) = \sum_{i=d}^{k} \frac{1}{k\mu \frac{\binom{k-1}{d-1}}{\binom{i-1}{d-1}} - k\lambda}$$

Mean-field limit[5]

$$\mathbb{P}(T^{coc} > t) = \left(\frac{1}{\rho + (1-\rho)e^{t\mu(d-1)}} \right)^{\frac{\alpha}{\alpha-1}}$$

[5] M. Bramson, Yi Lu, B. Prabhakar, Randomized load balancing with general service time distributions. SIGMETRICS 2010: 275-286

Outline

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- Central Queue Architecture
- Order Independent Descriptor
  - Redundancy and cancel on complete
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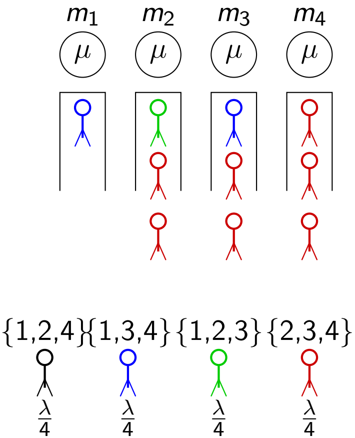
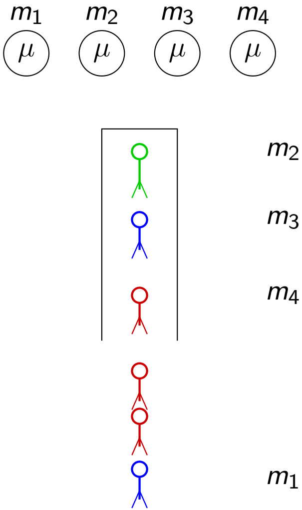
- Generalizations:
  - Token-based framework
  - Generalized Order Independent
- Impact of assumptions: scheduling and independence

Redundancy *c.o.s.* is not OI

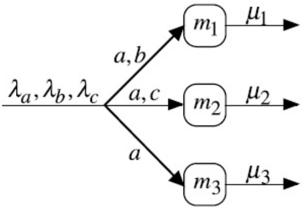
$N = 4$ *and* $d = 3$

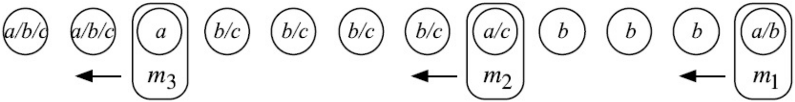




$(c_n, \ldots, c_1)$ no product form!

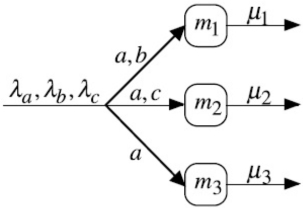
Multi-type job and multi-type servers[6]

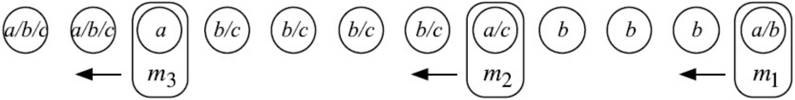




[6] Visschers, Adan, Weiss, *A product form solution to a system with multi-type jobs and multi-type servers, Queueing Systems, 2012.*

Multi-type job and multi-type servers[6]





- Markovian descriptor of (aggregated) form $s = (n_i, M_i, \ldots, n_d, M_d, \ldots, M_1)$
- If i servers are busy, then departure rate is $i\mu$
- Assignment rule determines server in case multiple compatible servers are available

[6]Visschers, Adan, Weiss, *A product form solution to a system with multi-type jobs and multi-type servers, Queueing Systems, 2012.*

Multi-type job and multi-type servers (cont.)

Key results: Existence of assignment rule, stability condition, characterization of steady-state distribution

$$\pi(\mathfrak{s}) = \alpha_i^{n_i} \cdots \alpha_1^{n_1} \frac{\Pi_\lambda(\{M_1, \ldots, M_i\})}{\Pi_\mu(M_i, \ldots, M_1)} \pi(0).$$

Caveat: No efficient way to calculate $\pi(0)$

[7] Adan, Weiss, A skill based parallel service system under FCFS-ALIS: steady state, overloads, and abandonments Stochastic Systems, INFORMS, 2014, 4, 250-299

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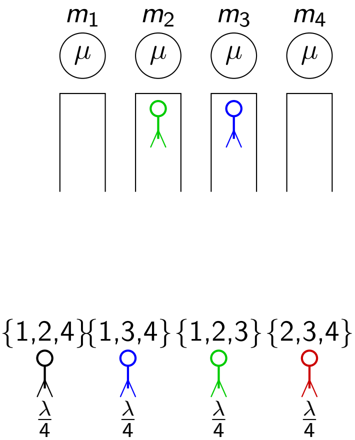
Results of similar nature with ALIS[7]

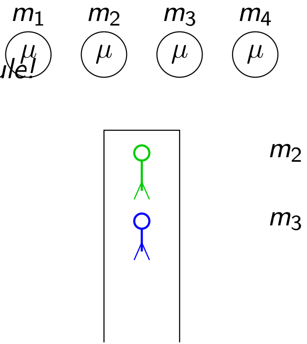
[7] Adan, Weiss, A skill based parallel service system under FCFS-ALIS: steady state, overloads, and abandonments Stochastic Systems, INFORMS, 2014, 4, 250-299

redundancy-d with $c.o.s.$ (example)

$N = 4$ and $d = 3$.

Uniform assignment satisfies assignment rule!





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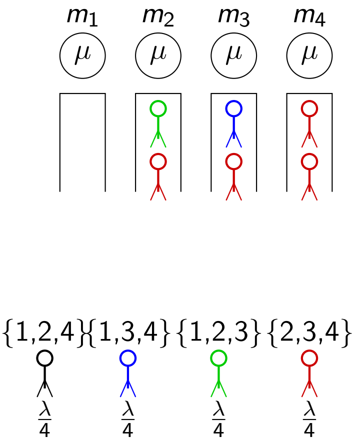

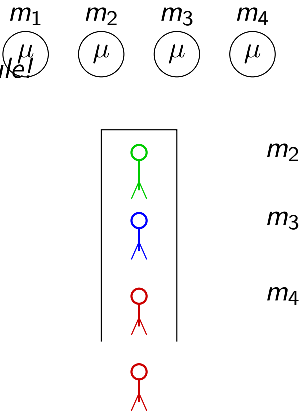

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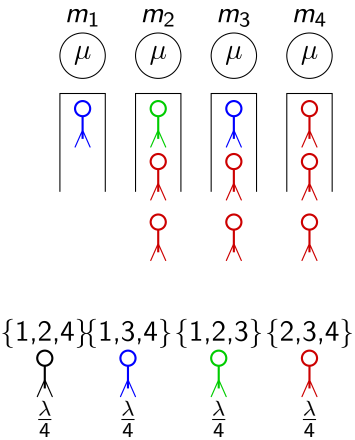
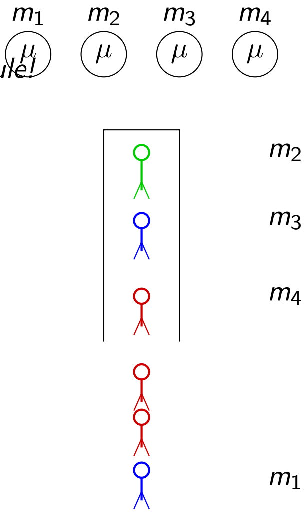
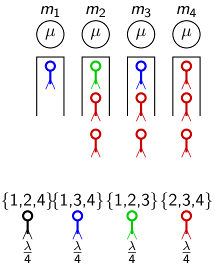


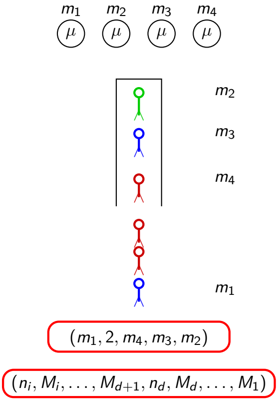


$$\boxed{(m_2, m_3, m_4, 2, m_1)}$$

$$\boxed{(M_1, \ldots, M_d, n_d, M_{d+1}, \ldots, M_i, n_i)}$$

redundancy-d with $c.o.s.$ – results





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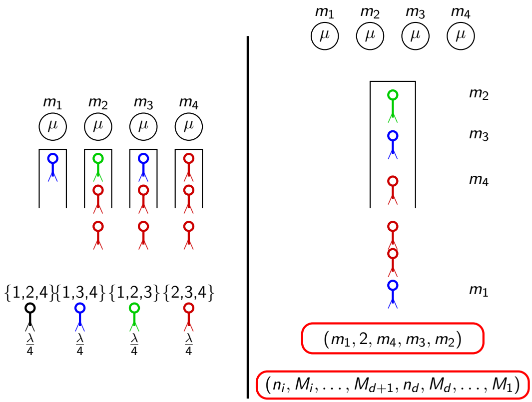


For any state $s = (n_i, M_i, \ldots, M_{d+1}, n_d, M_d, \ldots, M_1)$, we have

$$\pi(s) = {r_i}^{n_i} \ldots {r_d}^{n_d} \prod_{j=1}^{i} G_j(K, d) \frac{\pi(0)}{i!\mu^i} \rightarrow \text{product form}$$

redundancy-d with $c.o.s.$ – results[8]

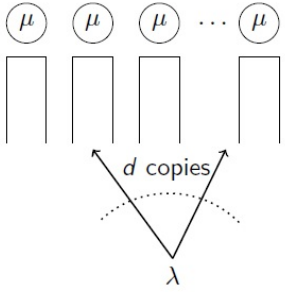
$$\pi(s) = {r_i}^{n_i} \dots {r_d}^{n_d} \prod_{j=1}^{i} G_j(K, d) \frac{\pi(0)}{i!\mu^i}$$

With a bit of algebra, and using the form of $\pi(s)$, we obtain

- $\pi(0)$, the normalizing constant
- $p(i)$, probability of i busy servers
- The P.G.F. of the number of waiting jobs in the system
- $E(N)$ the expected number of jobs in the system

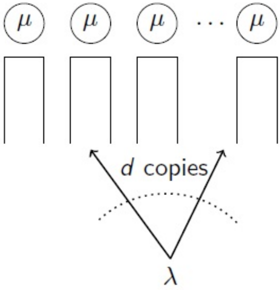
[8] A., Bodas, Verloop, On a unifying product form framework for redundancy models, Performance Evaluation, 2018

Comparing *c.o.s.* and *c.o.c.*



If $d = K$, c.o.c. is equivalent to an $M/M/1$ with rate μK
If $d = K$, c.o.s. is equivalent to an $M/M/K$

Comparing *c.o.s.* and *c.o.c.*

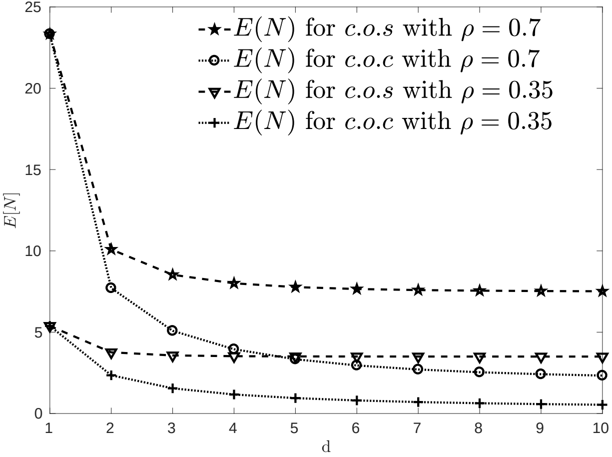


If $d = K$, c.o.c. is equivalent to an $M/M/1$ with rate μK
If $d = K$, c.o.s. is equivalent to an $M/M/K$

- What without i.i.d. assumption?

$d = K, c.o.c. \implies$ single server with rate μ

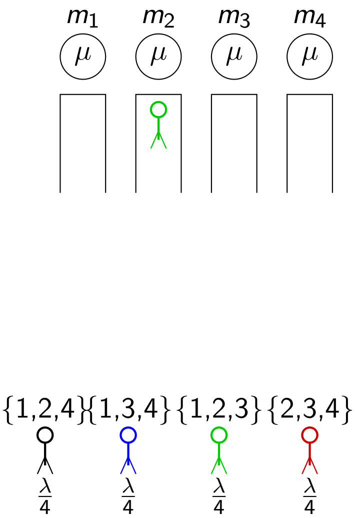
Performance comparison between *c.o.s.* and *c.o.c.*



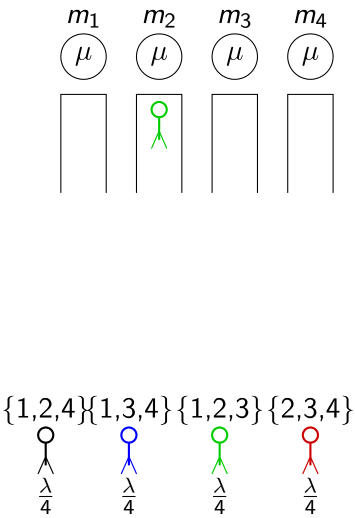
Sample path arguments

$N = 4$ *and* $d = 3$

redundancy-d with *c.o.s.*



JSW-d



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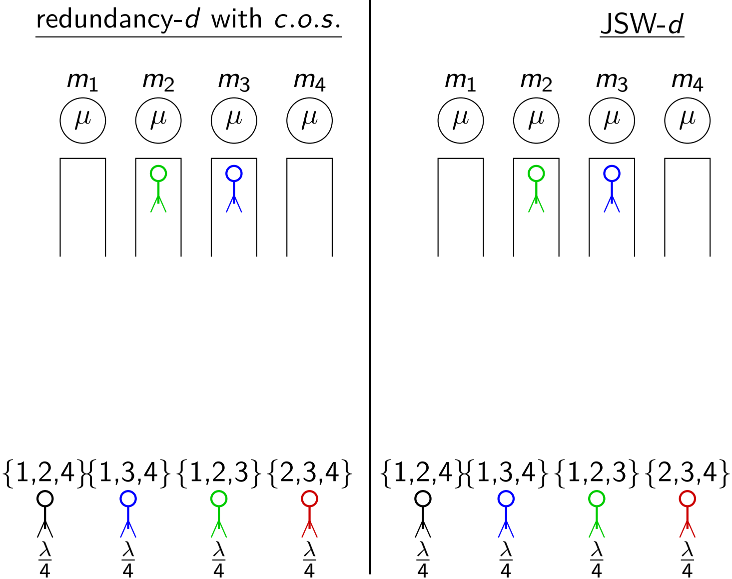

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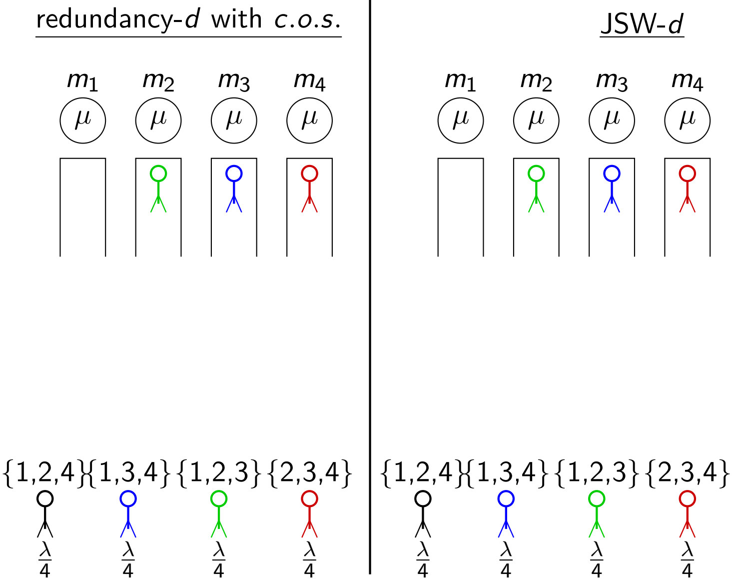

Sample path arguments

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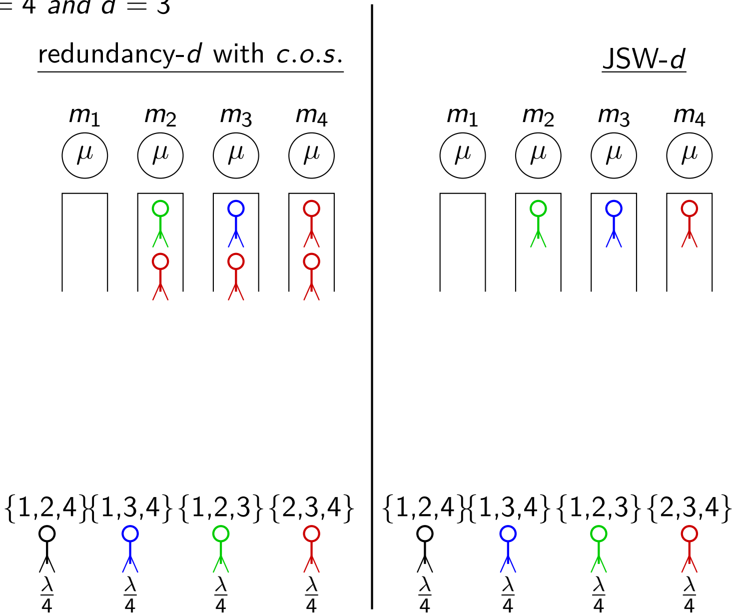
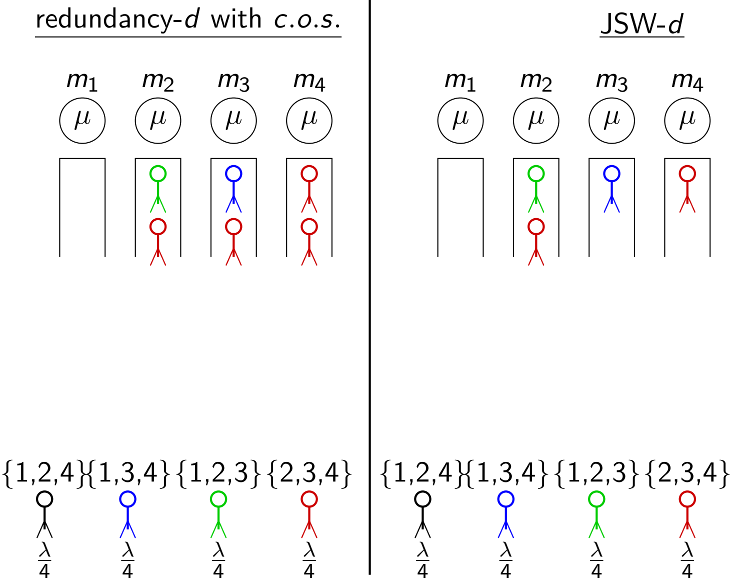
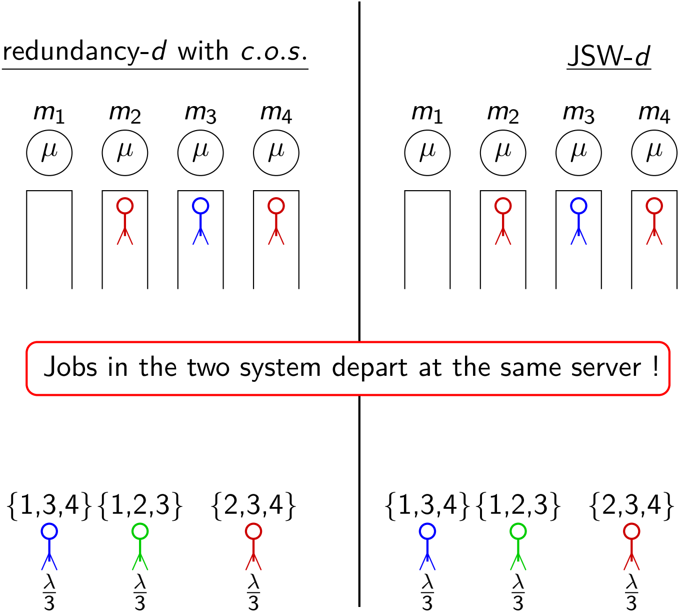


Sample path arguments

N = 4 *and d* = 3



Sample path coupling



Equivalence with JSW(d)[9]

Proposition

For any given sample-path realization, a given job will be served under both JSW(d) and redundancy-d with $c.o.s.$ in the same server.

We now know $\pi(0), p(i)$, P.G.F for the number of waiting jobs and $E(N)$ for JSW-d through the analysis of Redundancy c.o.s.

[9] A., Bodas, Verloop, On a unifying product form framework for redundancy models, Performance Evaluation, 2018

Redundancy with c.o.s. in mean-field[10]

$$\mathbb{P}(T > t) = \left(\lambda^d + (1-\lambda^d)e^{t(d-1)}\right)^{\frac{1}{d-1}}$$

$$\mathbb{E}(T^{cos}) = \sum_{n=0}^{\infty} \frac{\lambda^{dn}}{1 + n(d-1)}$$

[10] T. Hellemans, B. van Houdt, On the power-of-d-choices with least loaded server selection, Sigmetrics 2018

Redundancy with c.o.s. in mean-field[10]

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In the mean-field limit:

$$\mathbb{E}(T^{cos}) \geq \mathbb{E}(T^{coc})$$

[10] T. Hellemans, B. van Houdt, On the power-of-d-choices with least loaded server selection, Sigmetrics 2018

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  - Generalized Order Independent
- Impact of assumptions: scheduling and independence

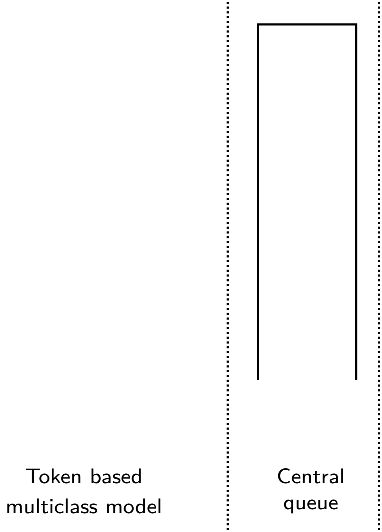
Token based framework[11]

- Multiclass jobs and tokens for service
- Product form stationary distribution
- Subsumes OI and Visschers et al., and more

[11] A., Bodas, Dorsman, Verloop, A token-based central queue with order-independent service rates , to appear in OR

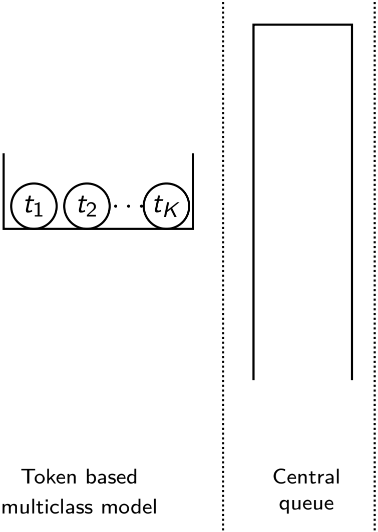
A token based central queue

A token based central queue



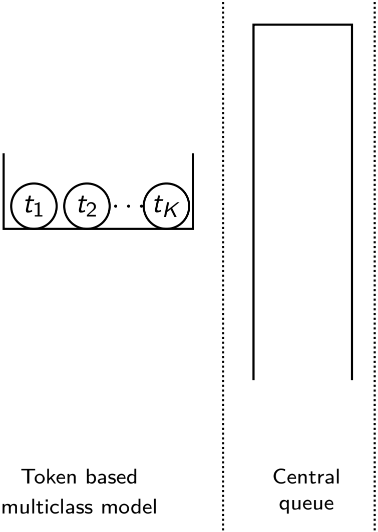
- A single central queue

A token based central queue



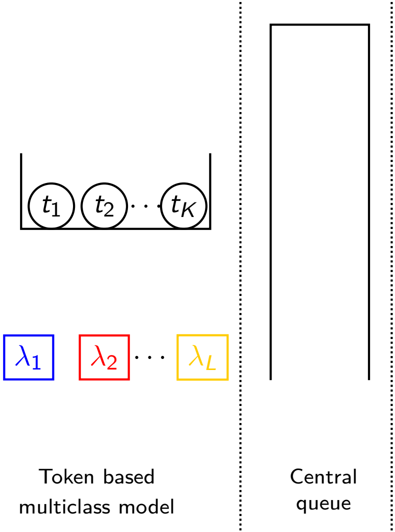
- A single central queue
- A set $\mathcal{T} = \{t_1, \ldots, t_K\}$ of K tokens

A token based central queue



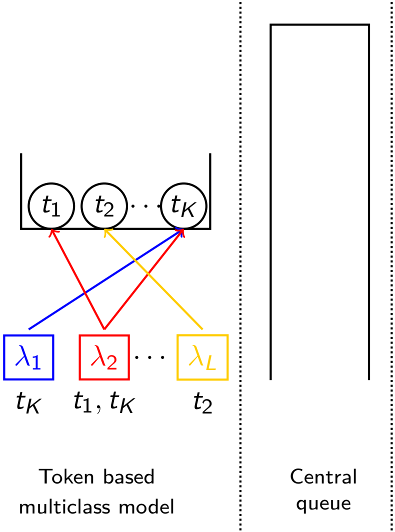
- A single central queue
- A set $\mathcal{T} = \{t_1, \ldots, t_K\}$ of K tokens
- Only jobs with tokens are served

A token based central queue



- A single central queue
- A set $\mathcal{T} = \{t_1, \ldots, t_K\}$ of K tokens
- Only jobs with tokens are served
- There are L job classes
- Poisson arrival rate λ_i for class i

A token based central queue



- A single central queue
- A set $\mathcal{T} = \{t_1, \ldots, t_K\}$ of K tokens
- Only jobs with tokens are served
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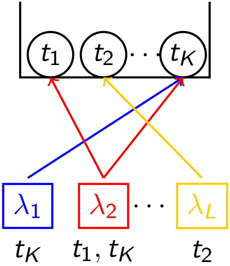
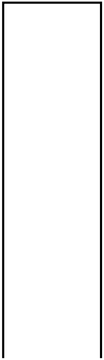
- A compatibility graph (arbitrary)

A token based central queue

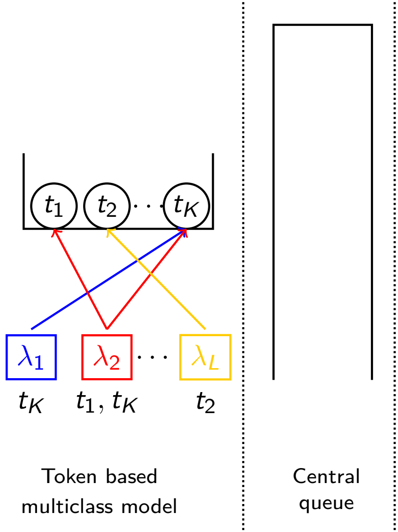


Token based multiclass model

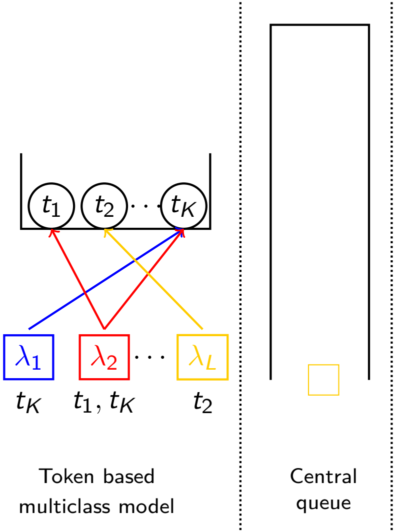


Central queue

- A single central queue
- A set $\mathcal{T} = \{t_1, \ldots, t_K\}$ of K tokens
- Only jobs with tokens are served
- There are L job classes
- Poisson arrival rate λ_i for class i
- A compatibility graph (arbitrary)
- Features of token based queue
  - Token assignment
  - Releasing a token
  - Token service rate
  - State space



- An arriving job must pick a compatible token if available



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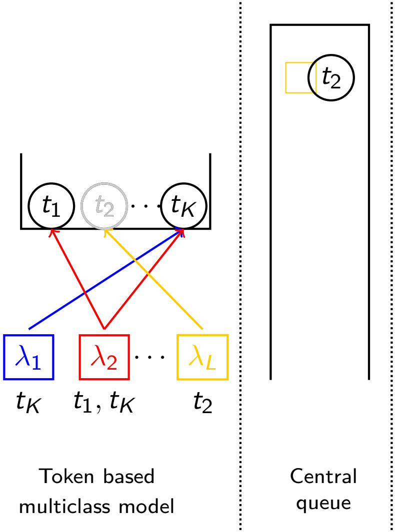

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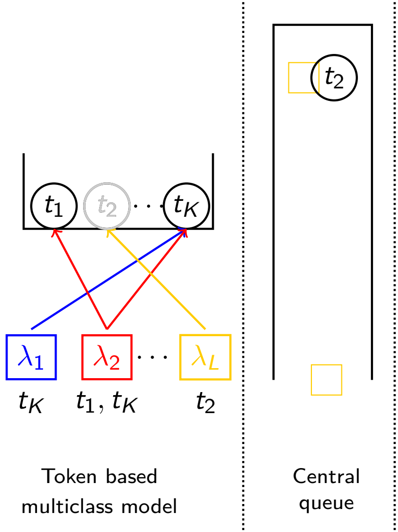

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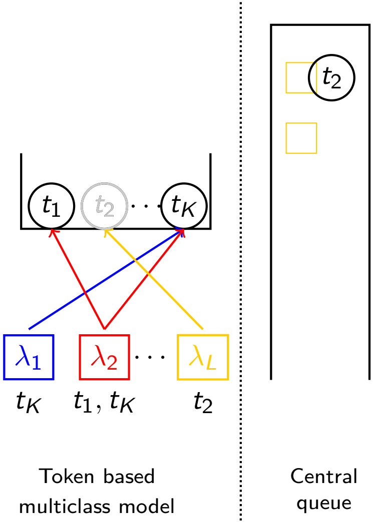

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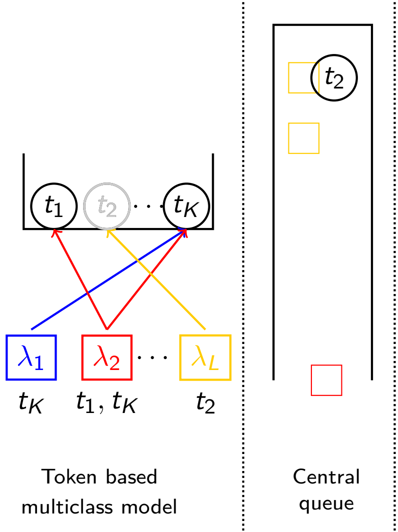

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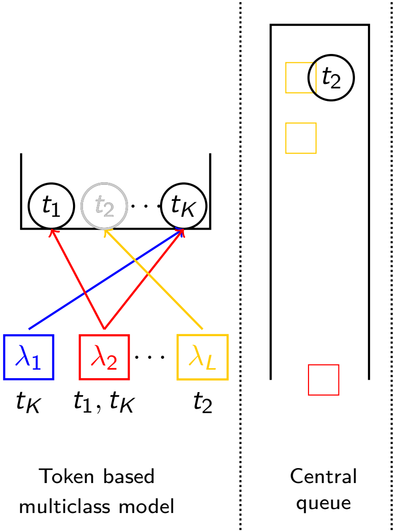

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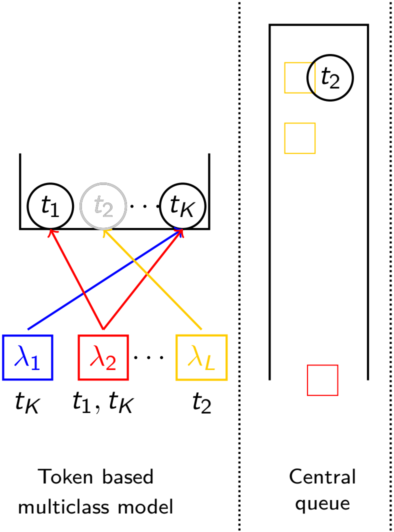

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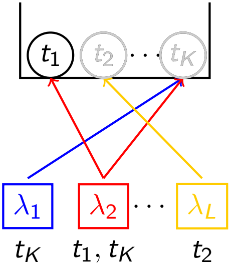
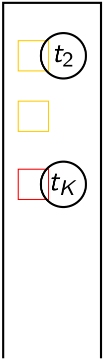


Token based multiclass model



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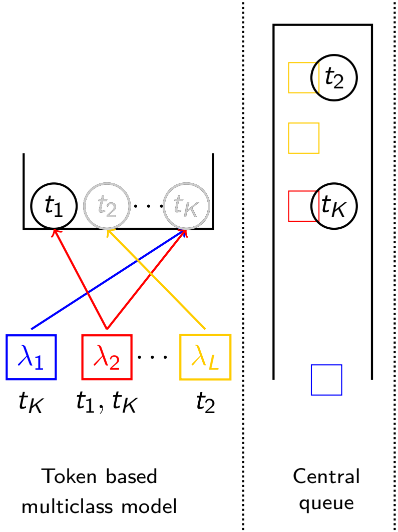

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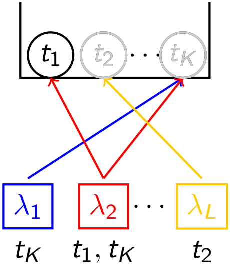

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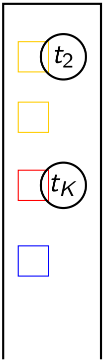

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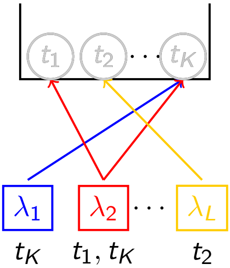

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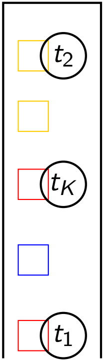

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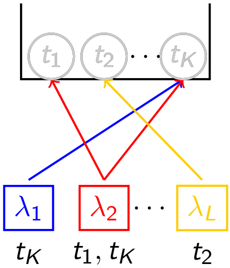

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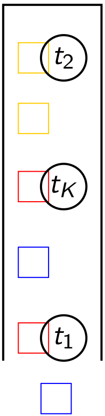

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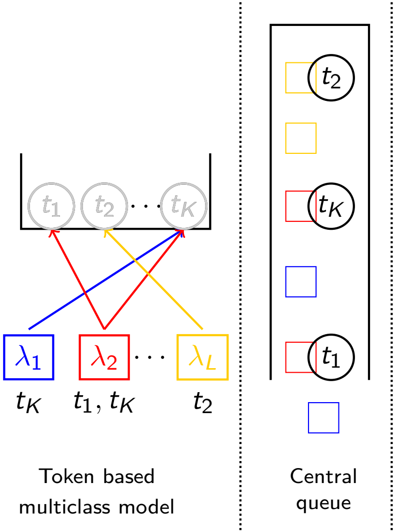

- Jobs have an exponential service requirement with unit mean
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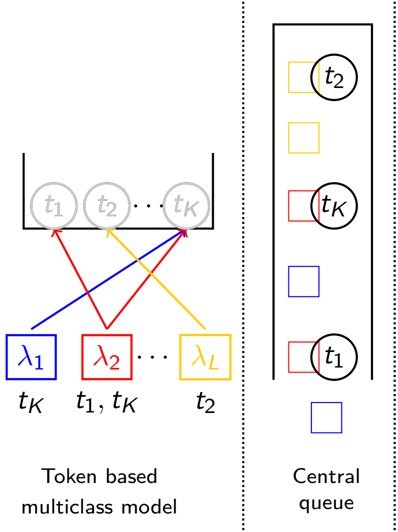

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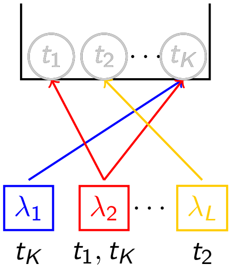

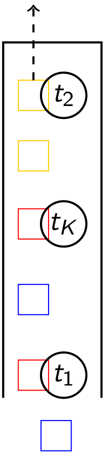

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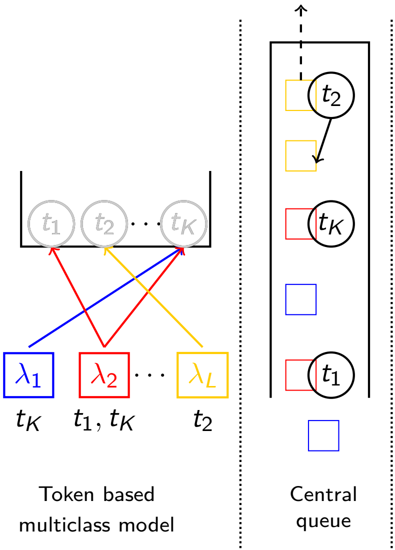

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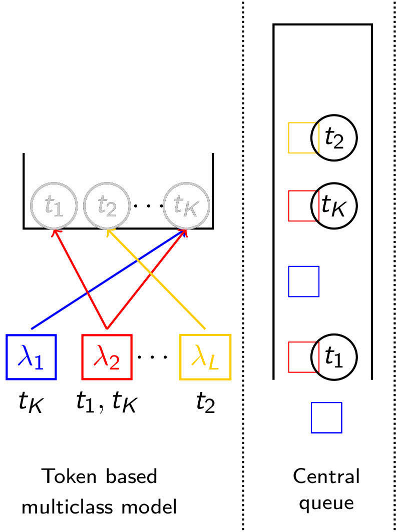

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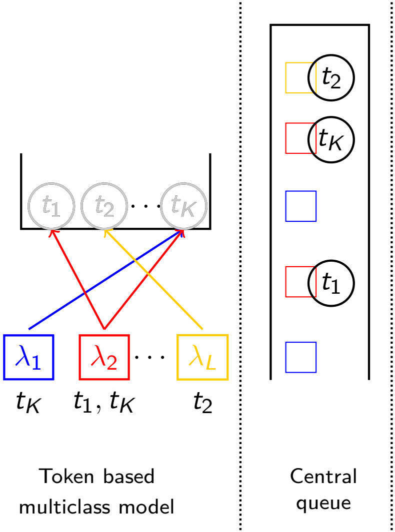

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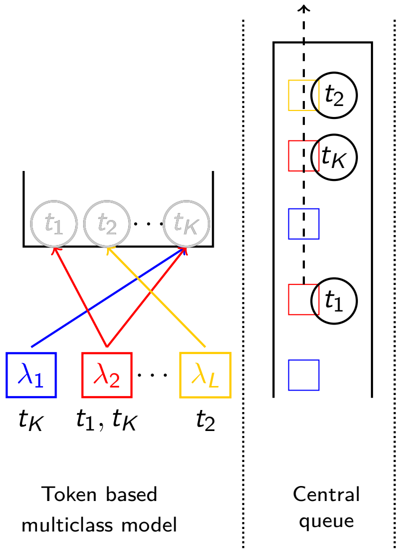

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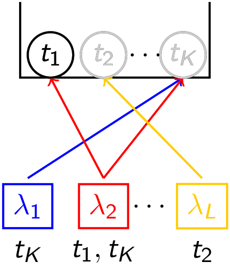
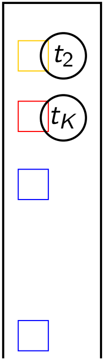


Token based multiclass model



Central queue

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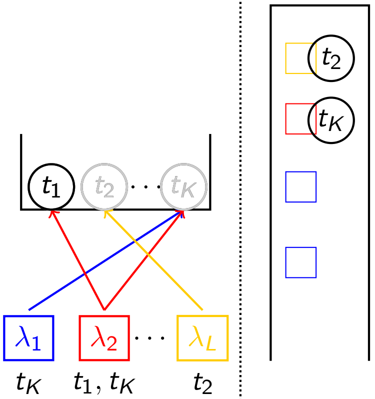

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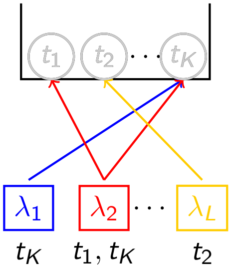

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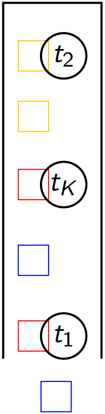
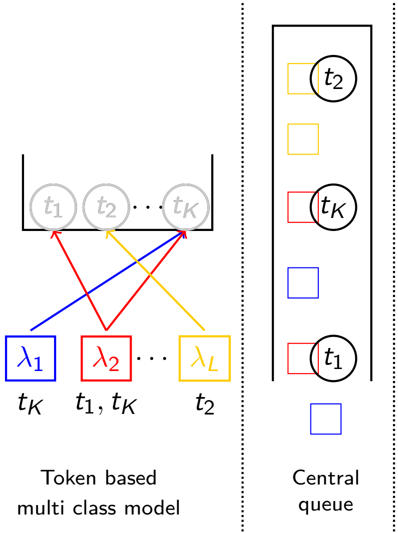


Central queue

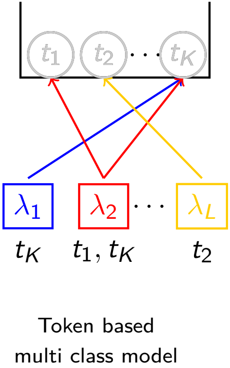
- Markovian descriptor for token based model
- Anonymize the class information
- $(t_2, 1, t_k, 1, t_1, 1)$
- $(T_1, n_1, \ldots, T_j, n_j, \ldots, T_i, n_i)$

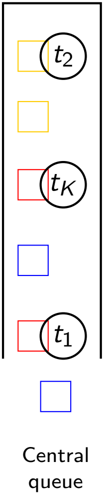
oldest job

jobs that are waiting for their compatible tokens

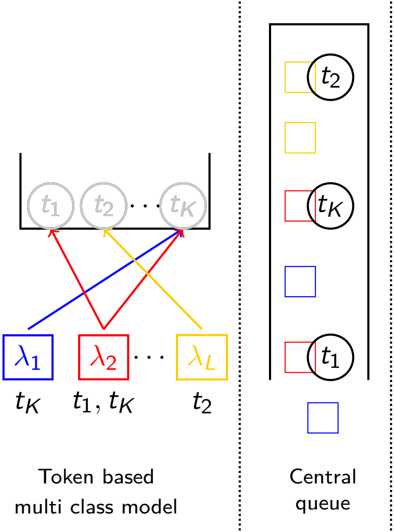


t_1 t_2 … t_K
λ_1 λ_2 … λ_L
t_K t_1, t_K t_2
Token based multi class model
t_2
t_K
t_1
Central queue





- Consider a generic state $x = (T_1, n_1, \ldots, T_j, n_j, \ldots, T_i, n_i)$



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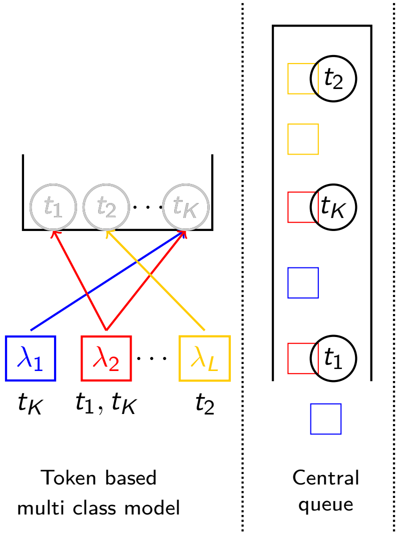

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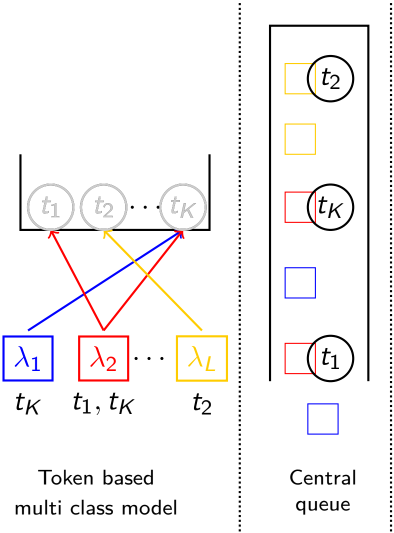

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Token service rate

Objects of departures T_j

$$s = (T_1, n_1, \ldots, T_j, n_j, \ldots, T_i, n_i)$$

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Require the departure function $\hat{\mu}(\cdot)$ satisfying the following

1. $\hat{\mu}(T_1 \ldots T_j) - \hat{\mu}(T_1 \ldots T_{j-1}) \geq 0$
2. $\hat{\mu}(T_1 \ldots T_j) = \hat{\mu}(T_{\sigma(1)} \ldots O_{\sigma(j-1)})$

Order Independent rates[12] $\Longrightarrow$ Sufficient condition for product-form

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For redundancy-d c.o.s., the T_j *correspond to server* M_j
For redundancy-d c.o.c., the T_j *correspond to first customer of a class* c_j

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State transitions

$$(T_1, n_1, .., T_j, n_j, .., T_i, n_i)$$

State transitions

$$\underbrace{\boxed{(T_1, n_1, .., T_j, n_j, .., T_i, n_i)}}_{x}$$

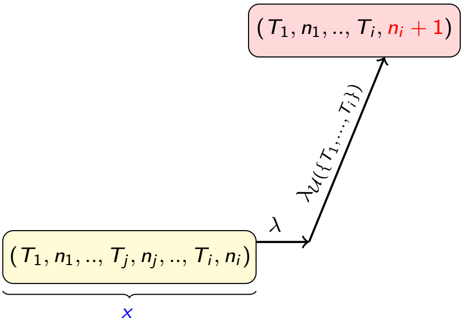
State transitions

$$\underbrace{\boxed{(T_1, n_1, .., T_j, n_j, .., T_i, n_i)}}_{x} \xrightarrow{\lambda}$$

State transitions

$$\begin{tikzcd} & (T_1, n_1, .., T_i, n_i + 1) \\ \underbrace{(T_1, n_1, .., T_j, n_j, .., T_i, n_i)}_{x} \arrow[ur, "\lambda"] & \end{tikzcd}$$

State transitions



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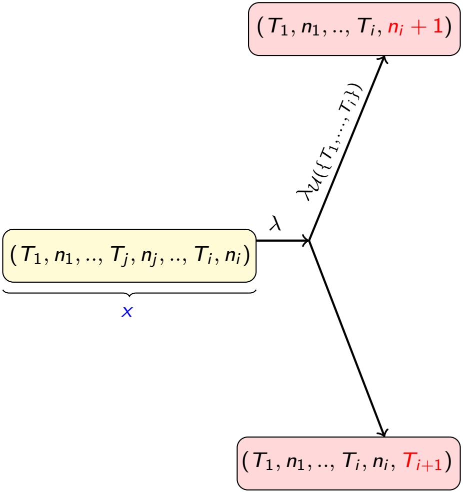

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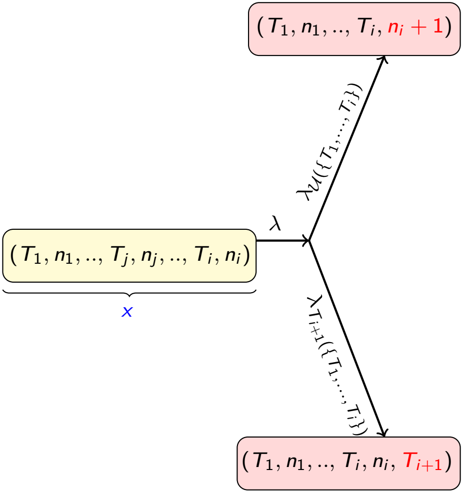

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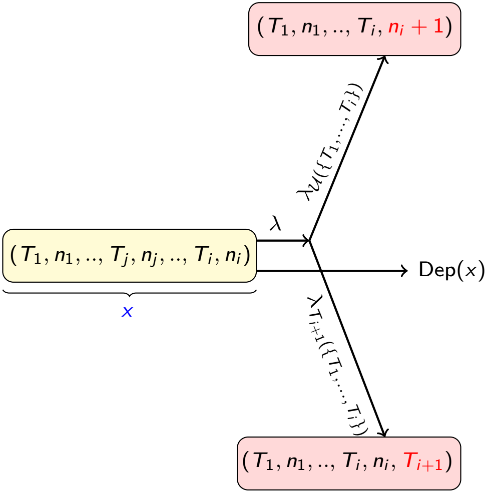

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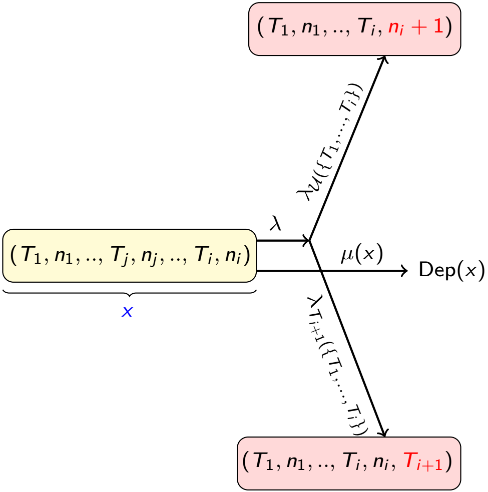

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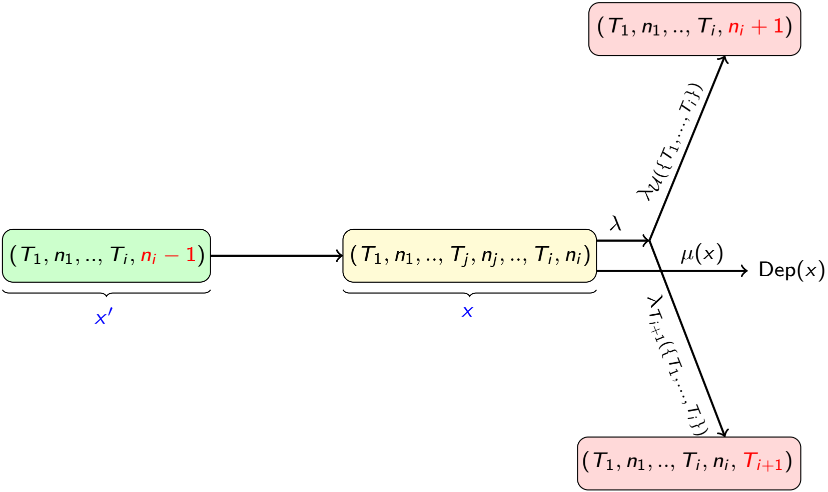

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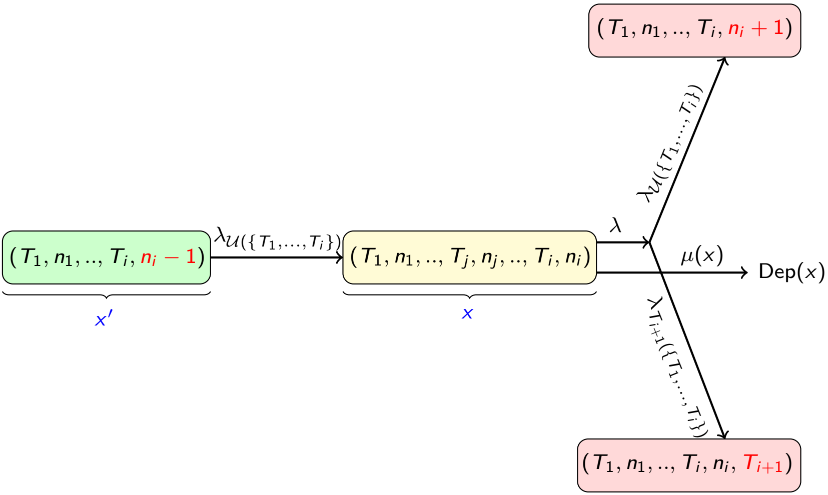

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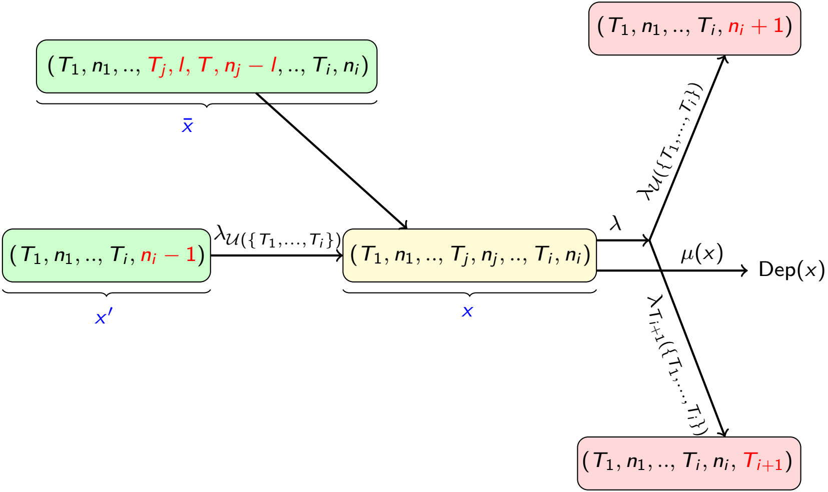

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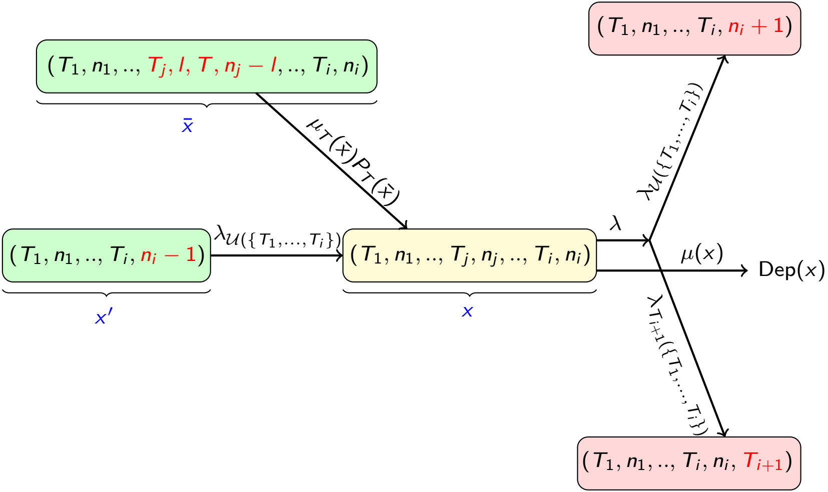

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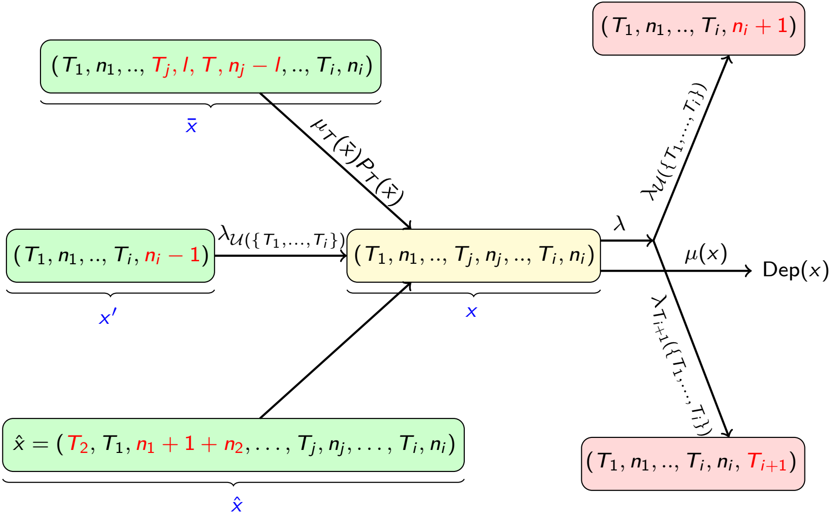

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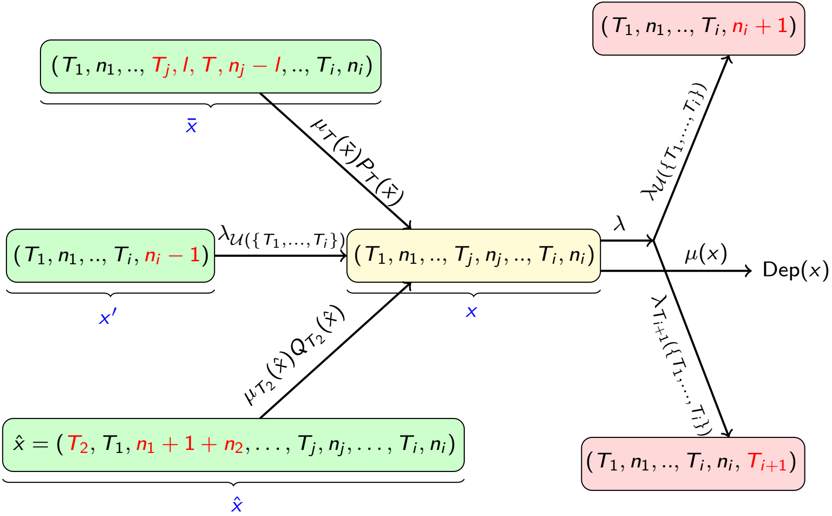
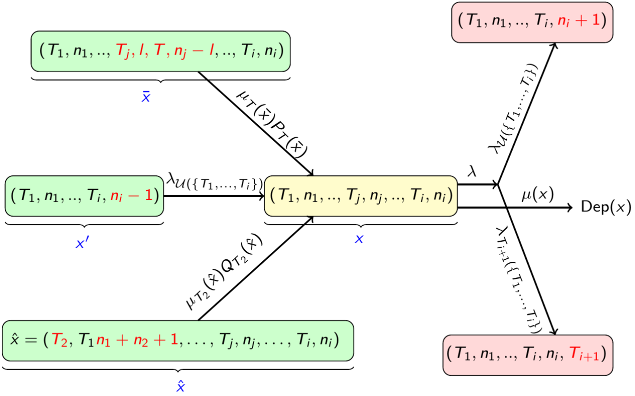


Global balance equations (GBE) for stationary distribution



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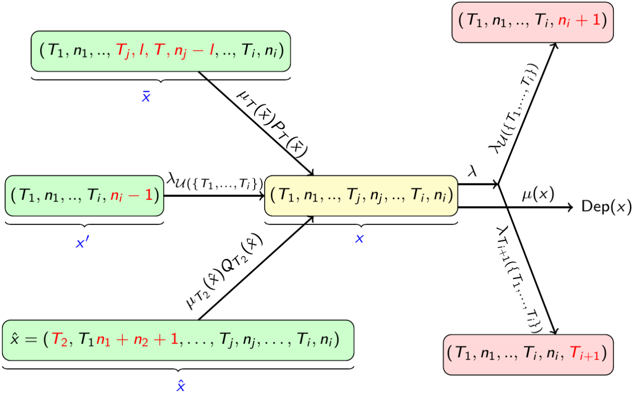

- GBE: Average rate of leaving state x = Average rate of entering state x

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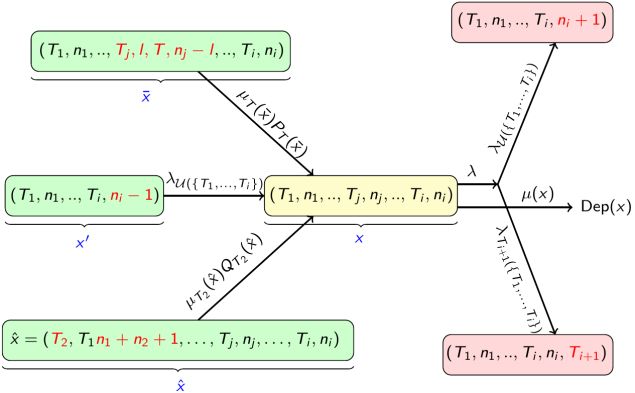

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$$\pi(x)\left[\lambda_{\mathcal{U}(\{\cdot\})} + \sum_{T_{i+1}} \lambda_{T_{i+1}}(\cdot) + \mu(x)\right]$$

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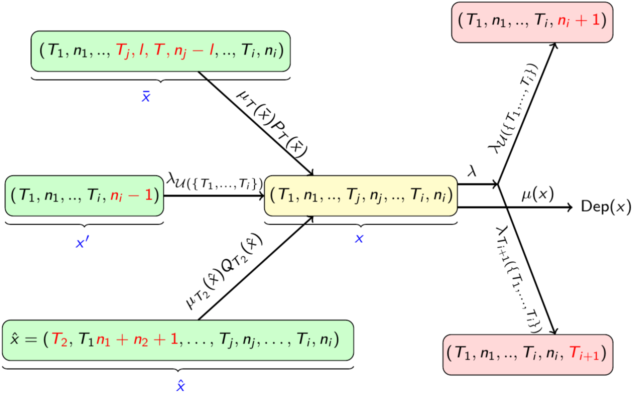

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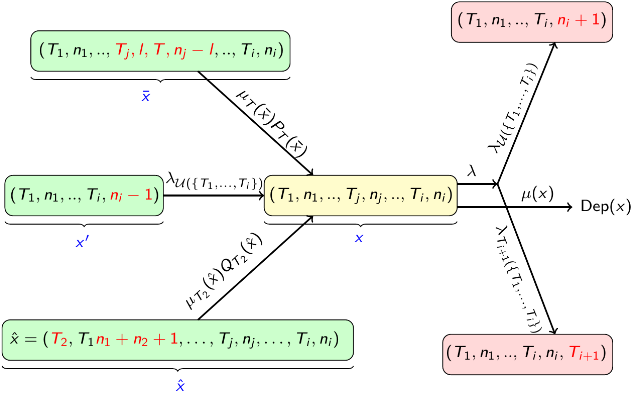

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- We solve this using partial balance equations (PBE's).

Main Result

Theorem

The steady state distribution for the token based model in state $x = (T_1, n_1, \ldots, T_j, n_j, \ldots, T_i, n_i)$ is given by

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$$\pi(x) = \pi(0) \ \prod_{j=1}^{i} \left(\frac{\lambda_{T_j}(\{T_1, \dots, T_{j-1}\})}{\sum_{l=1}^{j} \mu_{T_l}(x)} \right) \ \prod_{j=1}^{i} \left(\frac{\lambda_{\mathcal{U}(\{T_1, \dots, T_j\})}}{\sum_{l=1}^{j} \mu_{T_l}(x)} \right)^{n_j}$$

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normalising
constant

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- PGF for the number of waiting jobs and the total number of jobs

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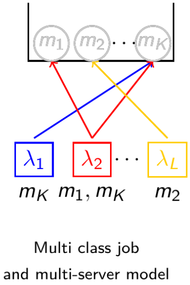
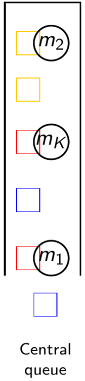
- PGF for the number of waiting jobs and the total number of jobs
- PGF for the waiting time and sojourn time for a job

Visschers and order-independent (OI) models





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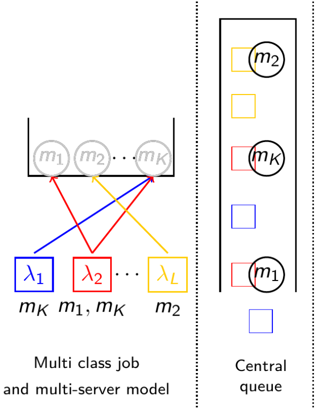

- Tokens replaced by servers

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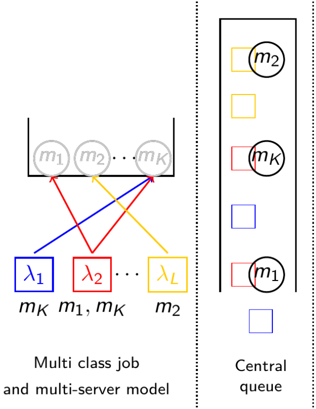

- Tokens replaced by servers
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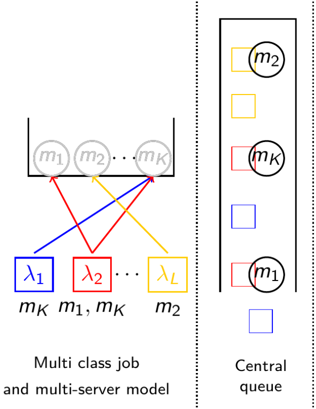

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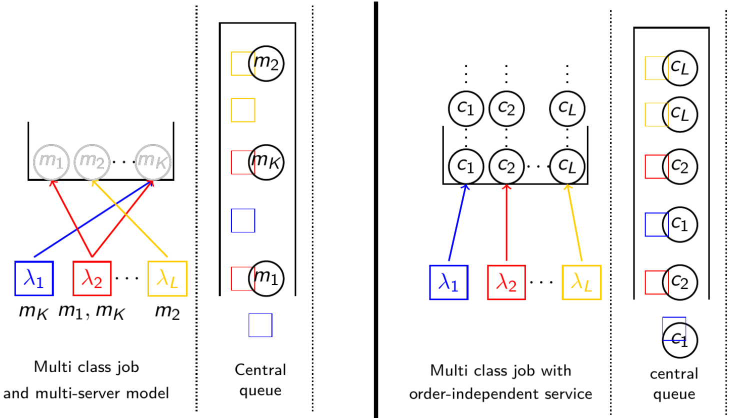

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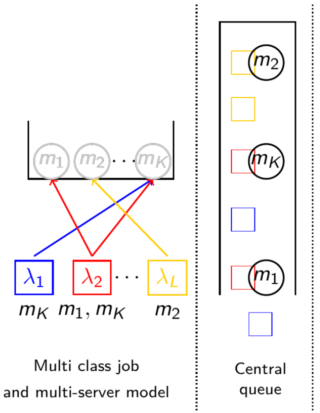

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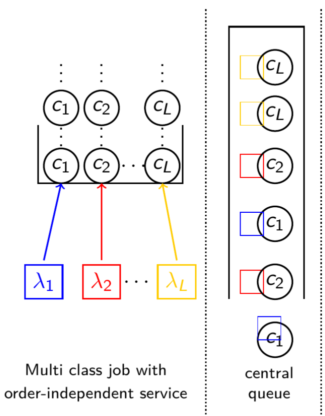
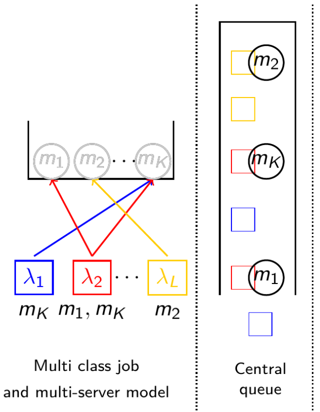
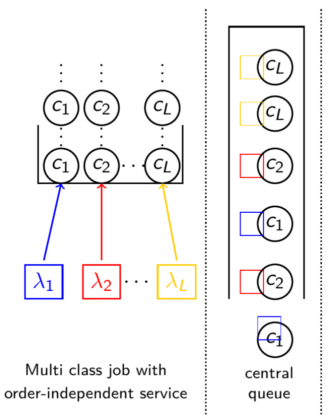


- Each job of class i gets a token c_i

Visschers and order-independent (OI) models





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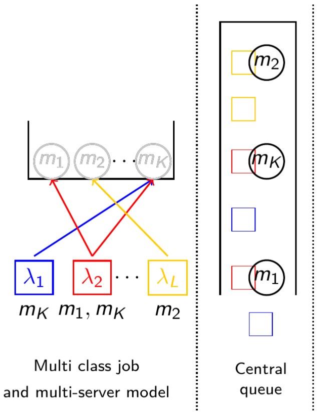
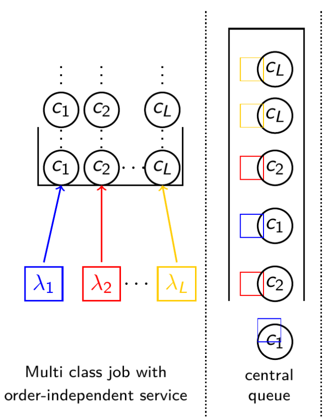
Visschers and order-independent (OI) models





- Tokens replaced by servers
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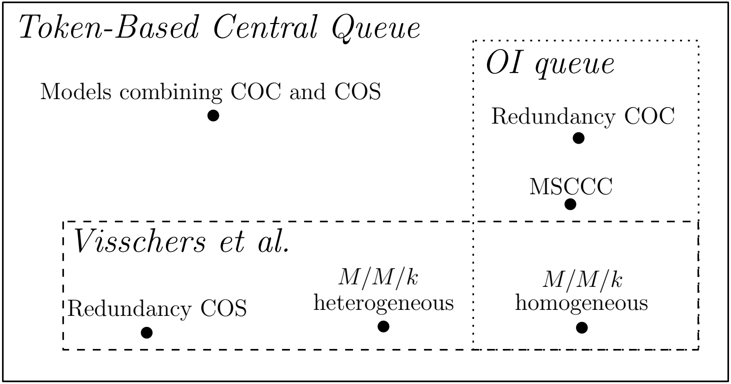
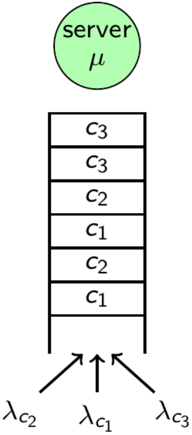
- No *assignment rule*

A classification of token-based central queues

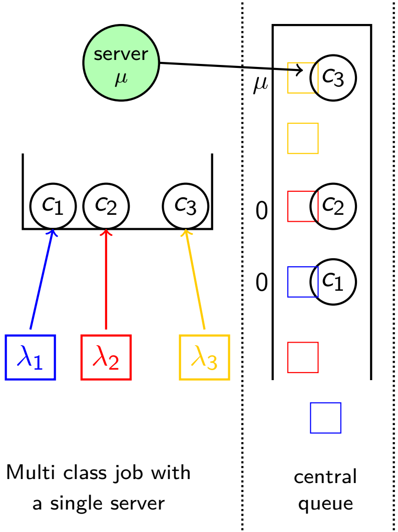


Example 1: Multiclass $M/M/1$ queue



- single server with multiple classes
- exponential service requirement
- state space of the form $(c_3, c_3, c_2, c_1, c_2, c_1)$
- $\pi(c_3, c_3, c_2, c_1, c_2, c_1)$
  $= (1-\rho)\rho_{c_3}^2 \rho_{c_2} \rho_{c_1} \rho_{c_2} \rho_{c_1}$
  where $\rho_i = \frac{\lambda_i}{\mu}$

Example 1: Multiclass $M/M/1$ queue (token view)



- We associate a unique token per class
- state space $(c_3, 1, c_2, c_1, 2)$
- Service rate μ for the first token
  service rate 0 for the rest

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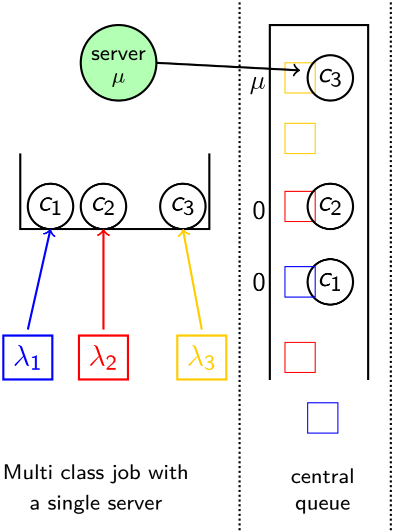

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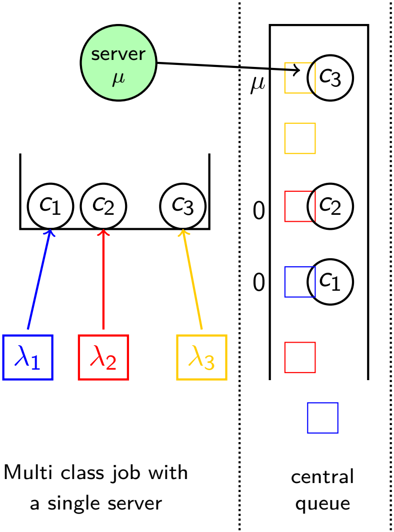

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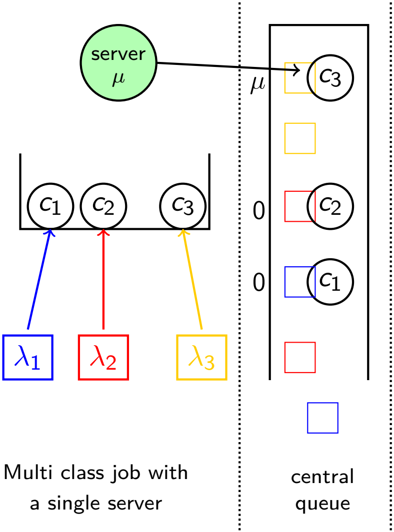

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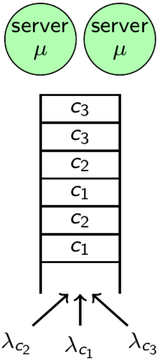
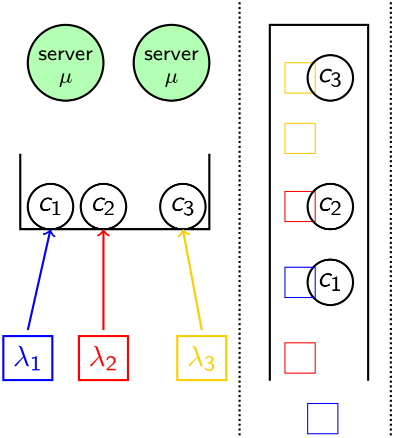
Example 2: Multi-server station with concurrent control of customers (MSCCC)



- 2 identical servers and 3 classes
- Servers are compatible with all classes
- Both servers cannot serve the same class at any time (concurrent control)

Example 2: Multi-server station with concurrent control of customers (MSCCC)



Multi class job with multiple server

central queue

- One token per class as earlier
- At most 1 job per class is served (concurrent control)
- Two server, three tokens
- Associate the two servers with the first 2 active tokens

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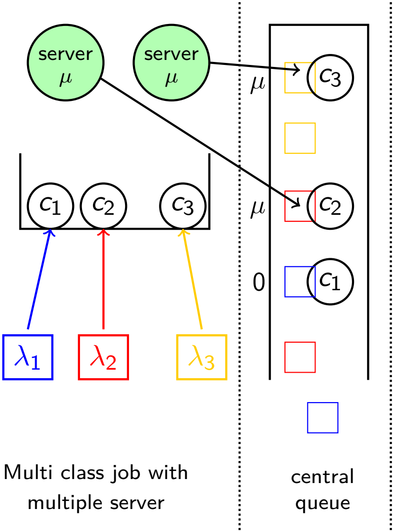

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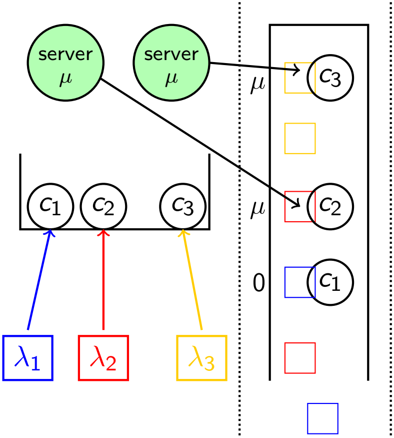

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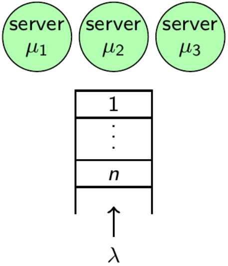
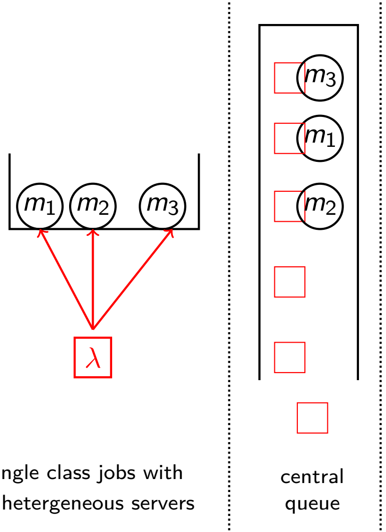
- $\pi(c_3, 1, c_2, c_1, 2) = (1-\rho)\left(\frac{\lambda_{c_3}}{\mu}\right)^2 \left(\frac{\lambda_{c_2}}{2\mu}\right) \left(\frac{\lambda_{c_1}}{2\mu}\right) \left(\frac{\lambda}{2\mu}\right)^2$

Example 3: $M/M/K$ queue



- One class of jobs and K possibly heterogeneous servers
- At most 3 jobs can be served at a time
- Also known as *Erlang-K* model
- When servers are identical, state space denoted by number of jobs in the system
- $\pi(n) = \pi(0)\frac{\rho^n K^K}{K!}$ where $\rho = \frac{\lambda}{K\mu}$

Example 3: $M/M/K$ queue



single class jobs with
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central
queue

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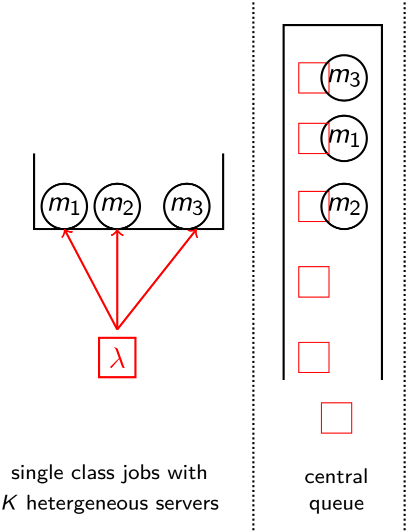

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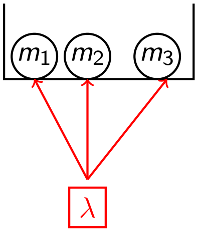

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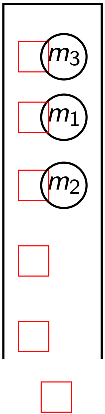

central
queue

- Single class & token m_i for server i
- Arriving jobs are compatible with all tokens

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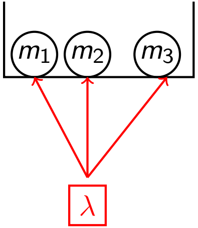

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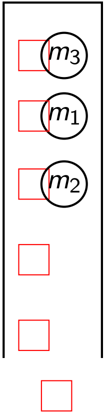

central
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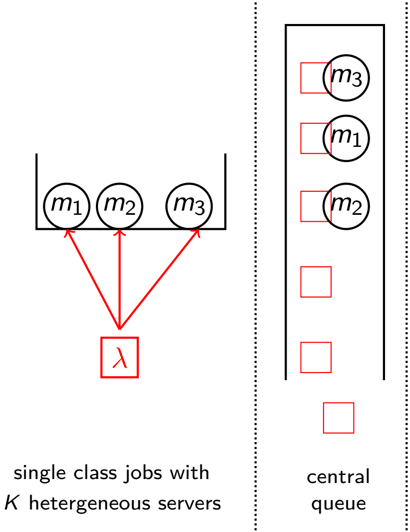

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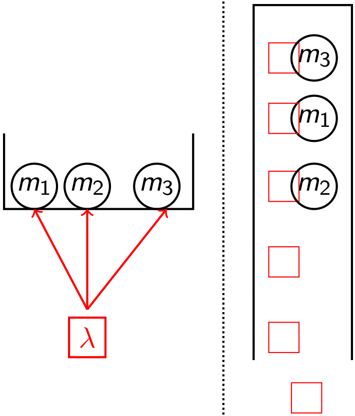

single class jobs with
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- $(m_3, m_1, m_2, 3)$
- $\pi(m_3, m_1, m_2, 3) =$

$$\pi(0)\frac{\lambda}{3\mu_3}\frac{\lambda}{2(\mu_3+\mu_1)}\frac{\lambda}{(\mu_3+\mu_1+\mu_2)}\left(\frac{\lambda}{(\mu_3+\mu_1+\mu_2)}\right)^3$$

Outline

- Redundancy
- Central Queue Architecture
- Order Independent Descriptor
  - Redundancy and cancel on complete
- Aggregated State Descriptor
  - Redundancy and cancel on start
- Generalizations:
  - Token-based framework
  - Generalized Order Independent
- Impact of assumptions: scheduling and independence

Generalized OI queue[13]

Detailed state descriptor $\vec{z}_m$, where m is number of jobs

- It includes jobs in service and in the queue

service $\Longrightarrow$ track server
queue $\Longrightarrow$ track class

[13] K. Gardner, R. Righter, Product Forms for FCFS Queueing Models with Arbitrary Server-Job Compatibilities: An Overview, to appear in QUESTA

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service $\Longrightarrow$ track server
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- $\mu(\vec{z}_m)$ satisfies the OI Properties

Theorem: The steady-state distribution of the Generalized OI queue is product form

$\Longrightarrow$ Unifying framework for product-form distributions

[13] K. Gardner, R. Righter, Product Forms for FCFS Queueing Models with Arbitrary Server-Job Compatibilities: An Overview, to appear in QUESTA

Outline

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Stability: Impact of independence assumption

- With i.i.d. copies and FCFS, redundancy does not impact stability.
  $\Longrightarrow$ papers by Gardner et al., Bonald et al.

$$d = K \quad \Longrightarrow \textit{single server with rate } \mu K$$
$$d = 1 \quad \Longrightarrow \quad K \textit{ indep. single servers with rate } \mu$$

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- What without i.i.d. assumption?

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Stability of redundancy: Impact of assumptions

Most of existing literature is with i.i.d. copies and FCFS
$\Longrightarrow$ stability not reduced
yield results that are qualitatively misleading.
Need of better models: $S\&X$[14]

[14] Gardner, Harchol-Balter, Scheller-Wolf, Van Houdt, A Better Model for Job Redundancy: Decoupling Server Slowdown and Job Size, IEEE/ACM ToN, 2017

Stability of redundancy: Impact of assumptions

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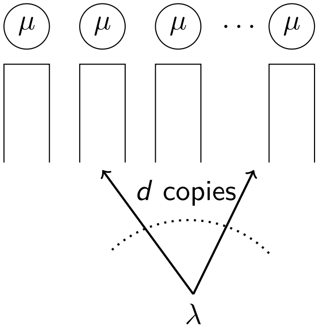
Need of better models: $S\&X$[14]

Main questions:

- How does redundancy with identical copies impact stability?
- Does stability depend on the scheduling discipline?

[14] Gardner, Harchol-Balter, Scheller-Wolf, Van Houdt, A Better Model for Job Redundancy: Decoupling Server Slowdown and Job Size, IEEE/ACM ToN, 2017

Redundancy d

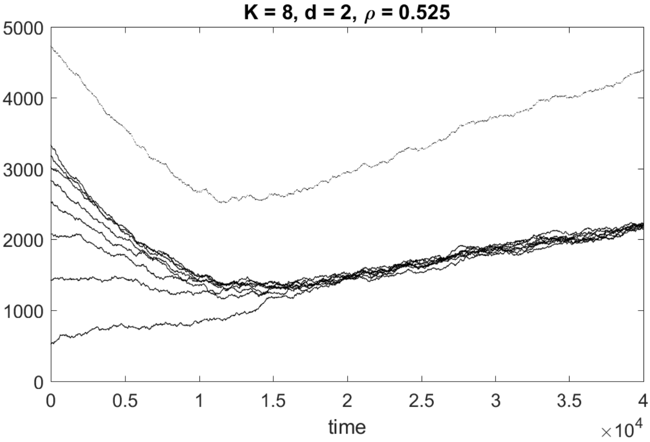


We study[15]:

- *c.o.c.* with identical copies
- scheduling discipline in servers can be PS, FCFS, or ROS.

[15] E. Anton, U. Ayesta, M. Jonckheere, I.M. Verloop. On the stability of redundancy models, to appear in OR

Instability with identical copies



Neither total number nor minimum are Lyapunov functions

Scheduling disciplines

- FCFS.
- Processor-Sharing (PS). The capacity is shared equally among all copies
- Random Order of Service (ROS)

[16] Baccelli, Foss, On the Saturation Rule for the Stability of Queues, JAP, 1995

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FCFS. Stability through saturated system[16]

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FCFS. Stability through saturated system[16]
For PS.

Lower Bound: *Every instant, put the copy with highest attained service, in the server with smallest number of copies.*
Upper Bound: *All copies need to be served for the job to be completed.*

[16] Baccelli, Foss, On the Saturation Rule for the Stability of Queues, JAP, 1995

Summary of stability conditions

	PS	FCFS	ROS	Priority policy
i.i.d	$\lambda < \mu K$	$\lambda < \mu K$	$\lambda < \mu K$	$\lambda << \mu K$
i.c.	$\lambda < \mu \frac{K}{d}$	$\lambda < \bar{\ell}\mu$	$\lambda < \mu K$	–

[17] Y. Raaijmakers, S. Borst, O. Boxma: Delta probing policies for redundancy. Perform. Eval (2018)

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i.c.	$\lambda < \mu \frac{K}{d}$	$\lambda < \bar{\ell}\mu$	$\lambda < \mu K$	–

- With PS, stability condition is the same as if all copies had to be served.
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Summary of stability conditions

	PS	FCFS	ROS	Priority policy
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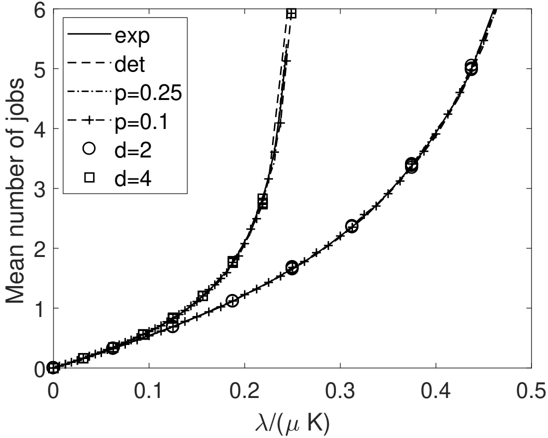
Need to develop strategies that preserve stability[17]

[17] Y. Raaijmakers, S. Borst, O. Boxma: Delta probing policies for redundancy. Perform. Eval (2018)

Non-exponential service requirements: PS[18]

Mean number of jobs with identical copies and exponential, deterministic and degenerate hyperexponential service requirements.



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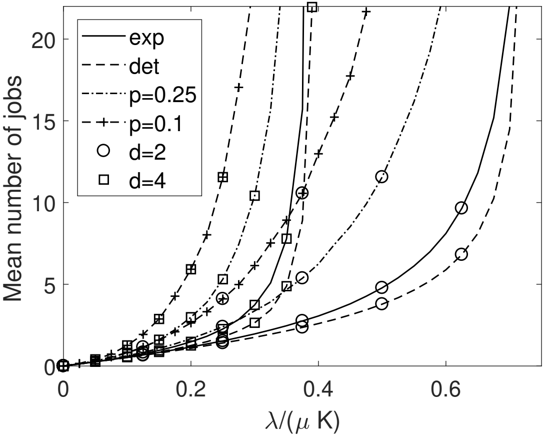

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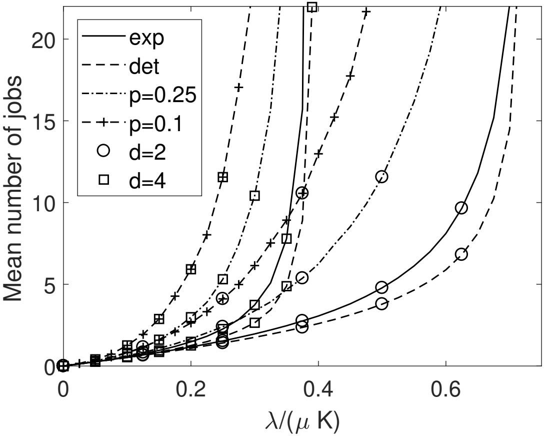

Approximations developed by: I Adan, M. Boon, G. Weiss, work in progress

What about heterogeneous systems? [19]

Assume $\mu_1 < \ldots < \mu_K$, redundancy-d.

- With redundancy, the stability condition is $\lambda^R = \min_{i=d,\ldots,K}\left\{\mu_i \frac{\binom{K}{d}}{\binom{i-1}{d-1}}\right\}$.

[19] E. Anton, U. Ayesta, M. Jonckheere, I. M.Verloop. Improving the Performance of Heterogeneous Data Centers through Redundancy, to appear in ACM SIGMETRICS 2021

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Redundancy-d has larger stability region than Bernoulli if $\mu_1 d < \mu_d$.

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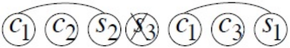
Relation with matching models

Model A:[20] :Arriving jobs wait in the queue for a compatible server, arriving servers match the first compatible job and leave the system (even if unmatched)



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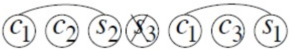

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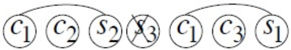
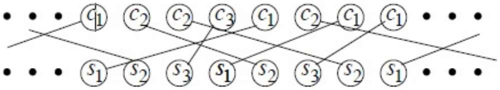


Same distribution as OI queues

Model B:[21] (FCFS infinite bipartite matching): arrivals are job server pairs, both of which can queue; arriving jobs (servers) match the first compatible server (job) if any, otherwise join queue



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Conclusions and Perspectives

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- Product form distribution does not directly imply computability
  $\Longrightarrow$ computation of $\pi(0)$.
- Not final word on product form distributions yet...
  $\Longrightarrow$ Pass & Swap queues (Comte& Dorsman 2020) have product form, yet not included in Token or Gen-OI

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  $\Longrightarrow$ E. Cardinaels, S. Borst, J.H. van Leeuwaarden, Redundancy Scheduling with Locally Stable Compatibility Graphs, arxiv 2020
- Relation between JSQ(d) and Redundancy
  $\Longrightarrow$ Coming up SNAPP talk by S. Borst

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Load balancing, redundancy, and multi type job and server systems

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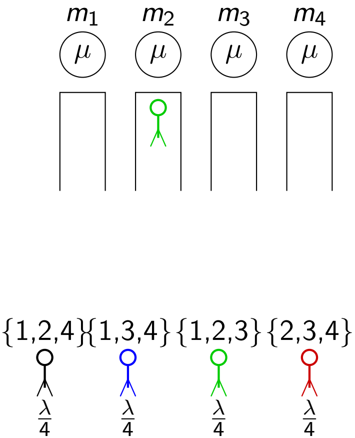
CNRS & Ikerbasque-Univ. Basque Country

Questions/Remarks welcome at : urtzi.ayesta@irit.fr

IFIP Performance 2020, 2/11/2020

redundancy-d with $c.o.s.$

$N = 4$ *and* $d = 3$



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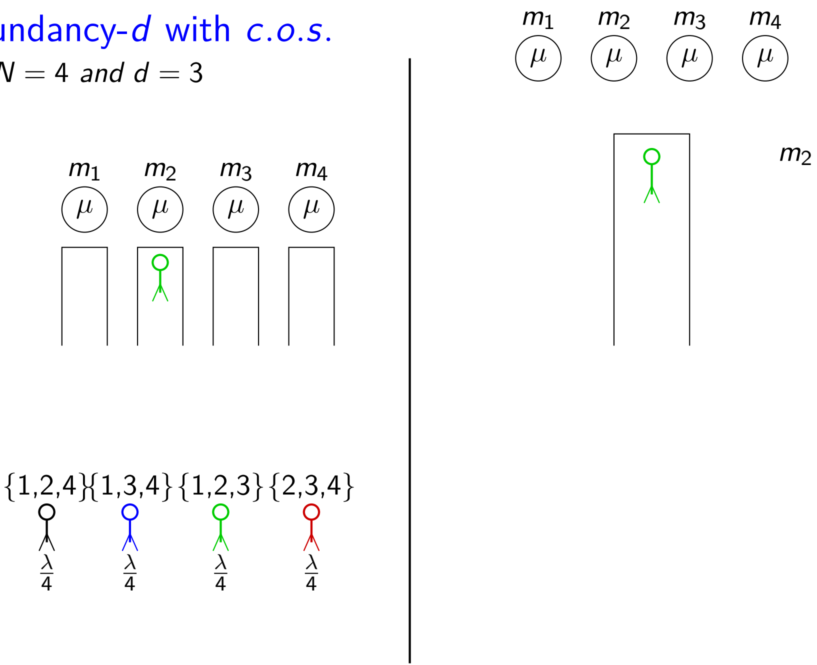

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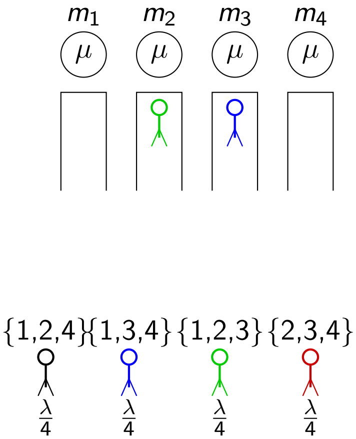

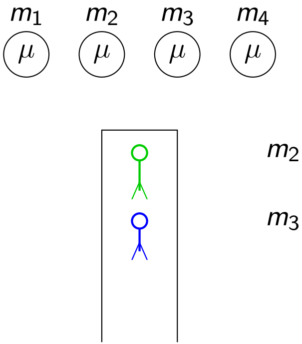

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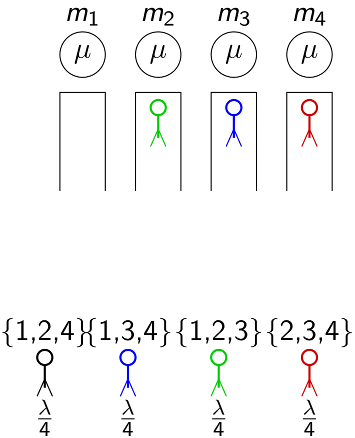

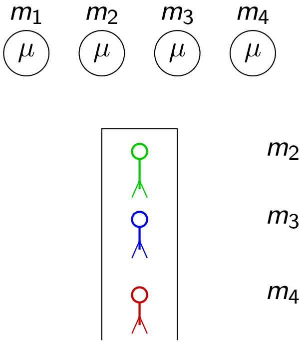

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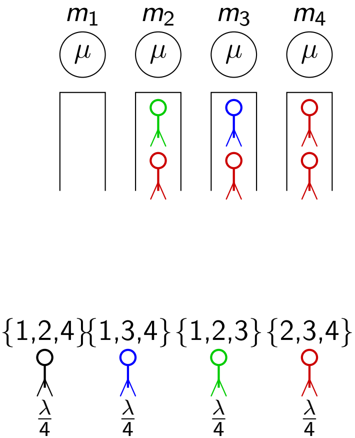

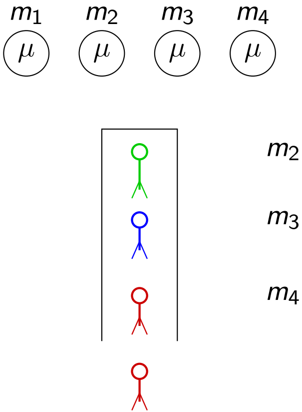

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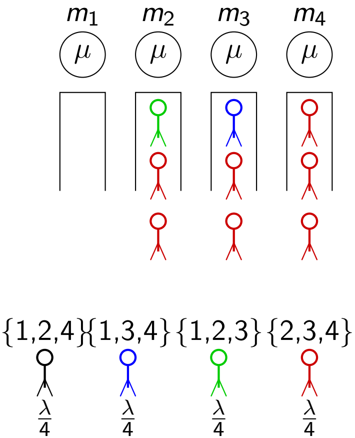

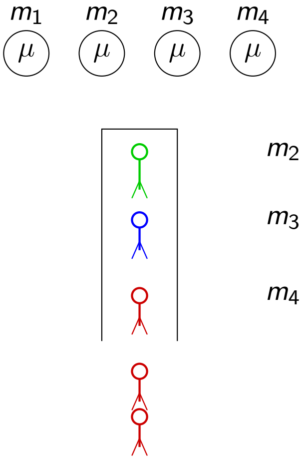

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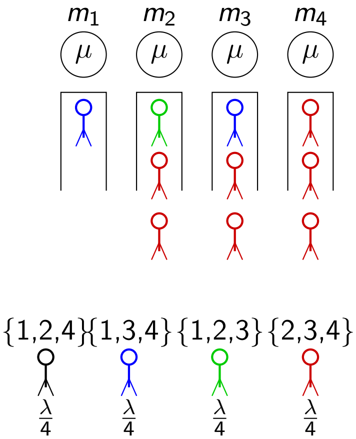

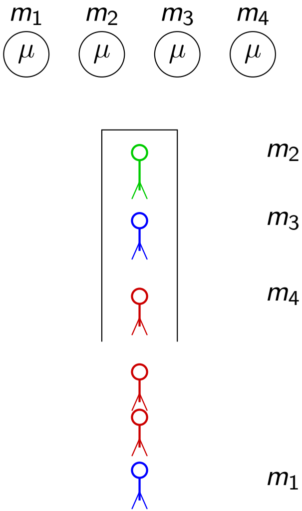

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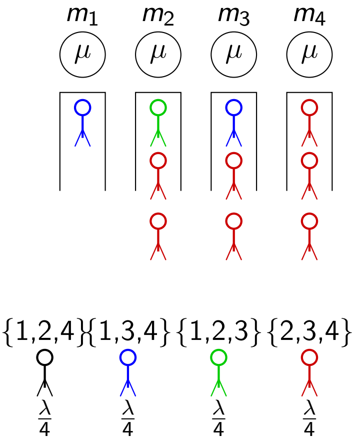

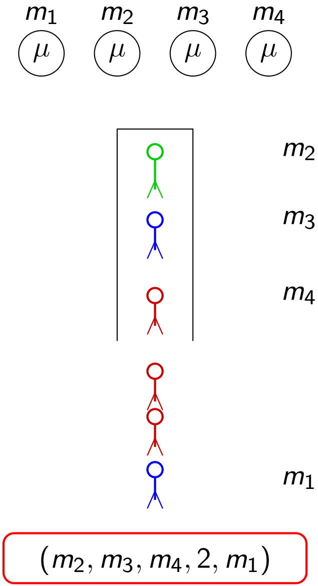

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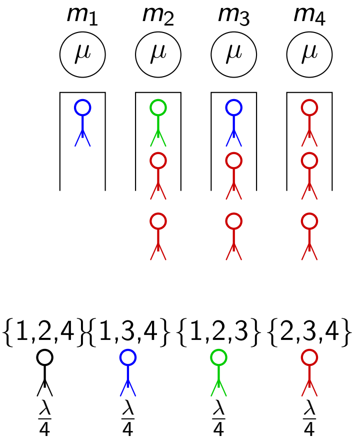
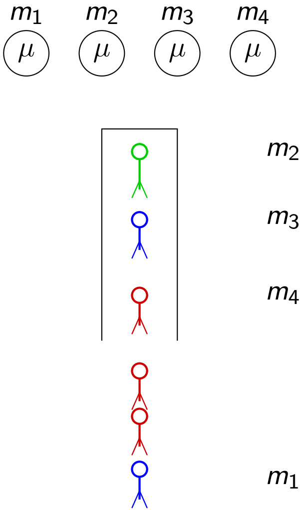
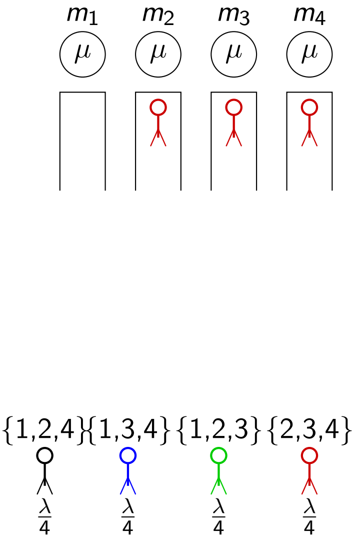




$$\boxed{(m_2, m_3, m_4, 2, m_1)}$$

$$\boxed{(M_1, \ldots, M_d, n_d, M_{d+1}, \ldots, n_i, M_i)}$$

redundancy-d with $c.o.c.$



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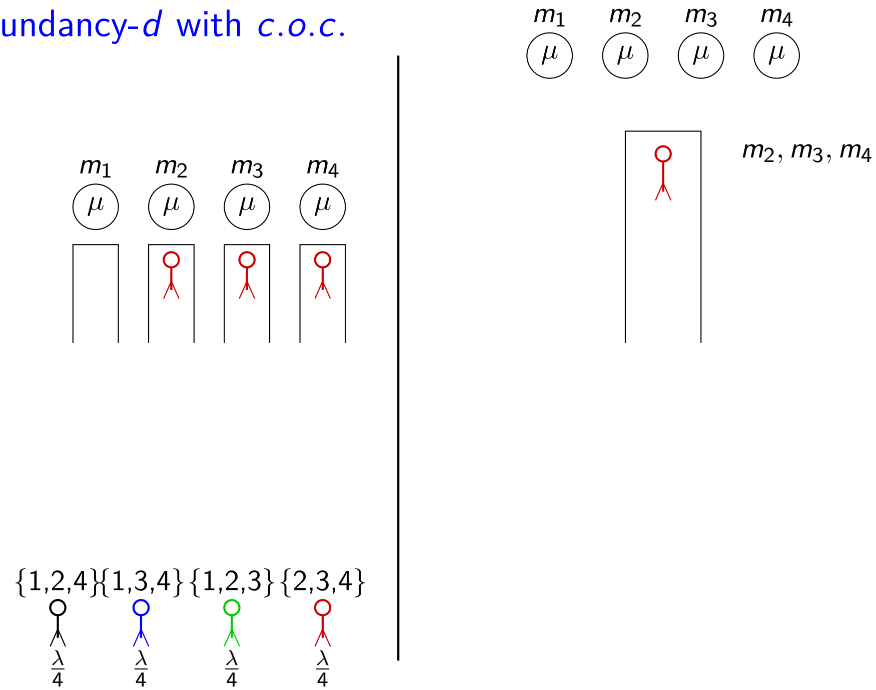

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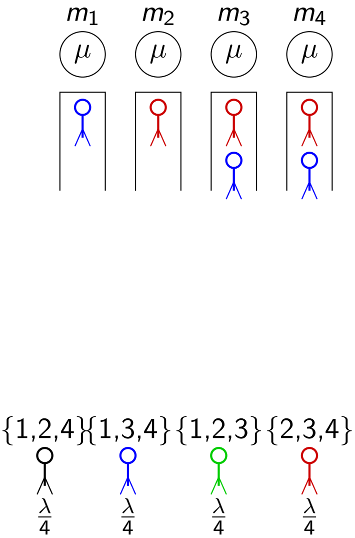

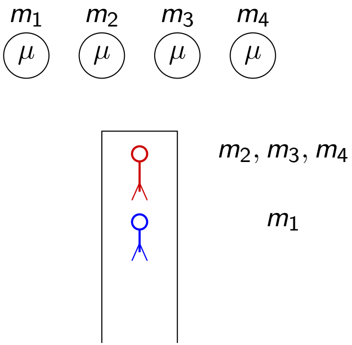

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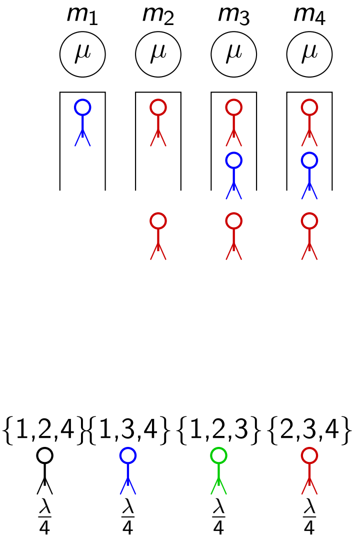

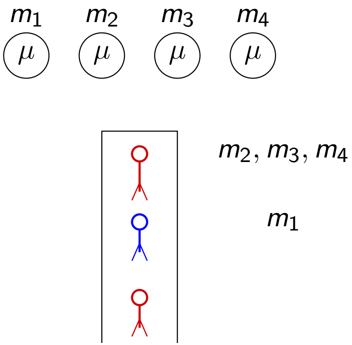

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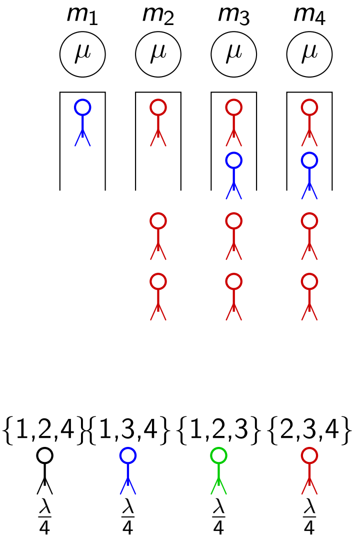

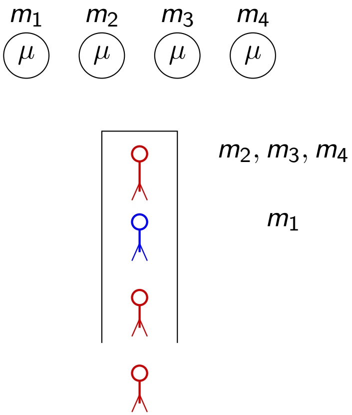

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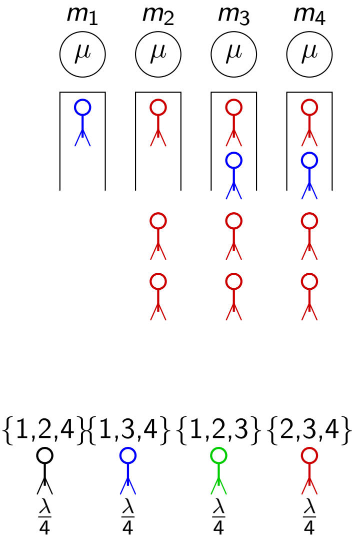
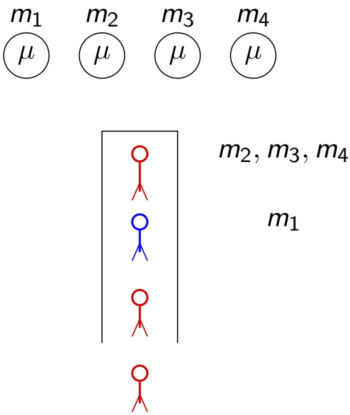
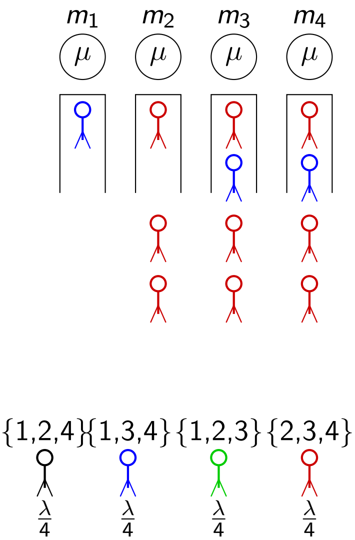
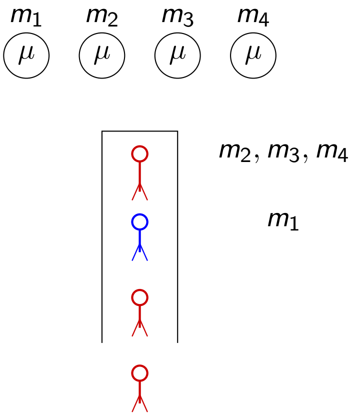




$(F_{red}, F_{blue}, 2)$

redundancy-d with $c.o.c.$

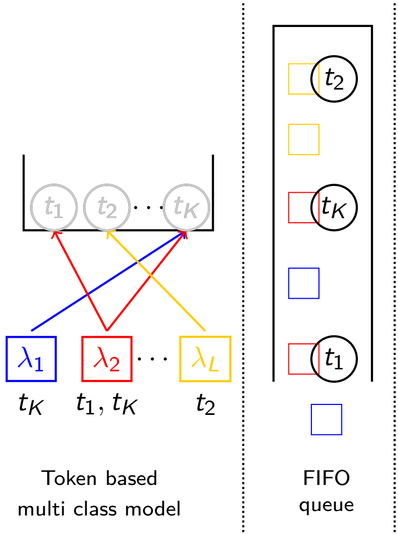




$$\boxed{(F_{red}, F_{blue}, 2)}$$

$$\boxed{F_{c_1}, n_1, F_{c_2}, n_2, \ldots, F_{c_i}, n_i}$$

Classes that wait



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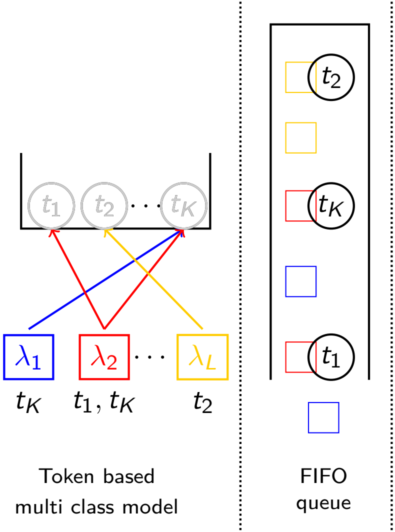

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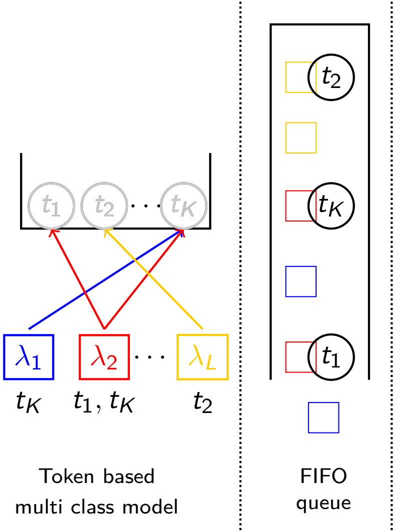
  $x = (T_1, n_1, \dots, T_j, n_j, \dots, T_i, n_i)$

Classes that wait



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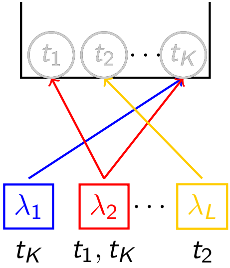
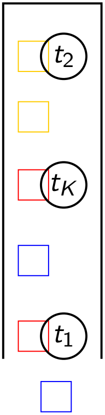
Classes that wait





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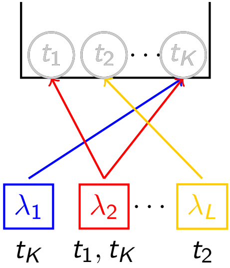
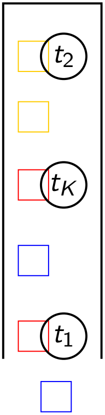
- What are the job classes that constitute the n_j jobs ?
- $\mathcal{U}(\{T_1, T_2, \ldots, T_j\})$ denotes the set of job classes who have to wait because tokens $\{T_1, T_2, \ldots, T_j\}$ are busy

Classes that wait



Token based
multi class model



FIFO
queue

- Consider a generic state $x = (T_1, n_1, \ldots, T_j, n_j, \ldots, T_i, n_i)$
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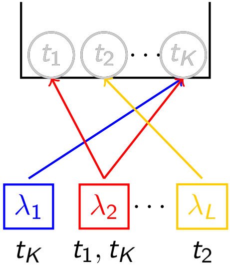

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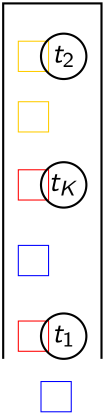

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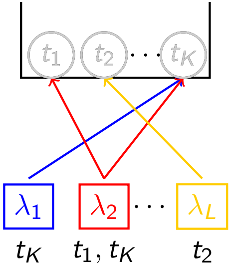

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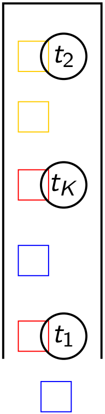

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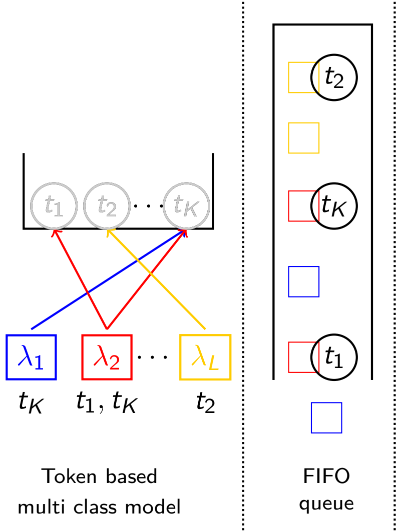
Activating a token



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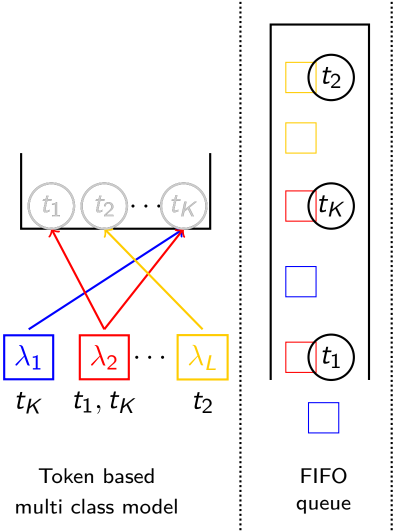

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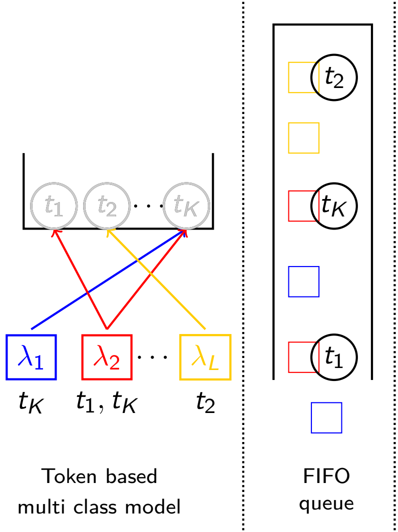

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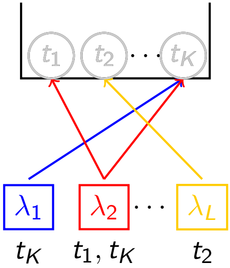
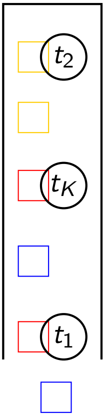


Token based
multi class model



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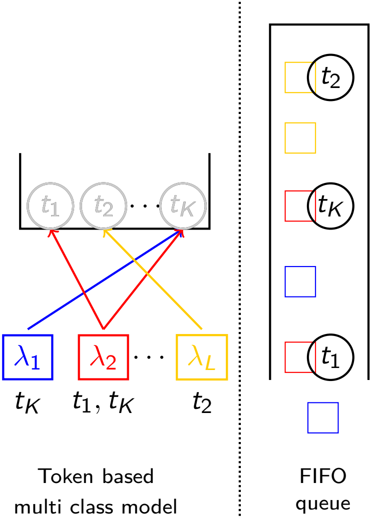

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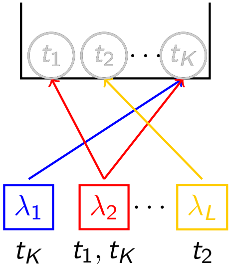

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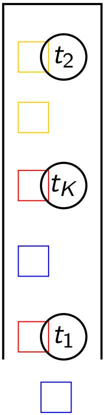

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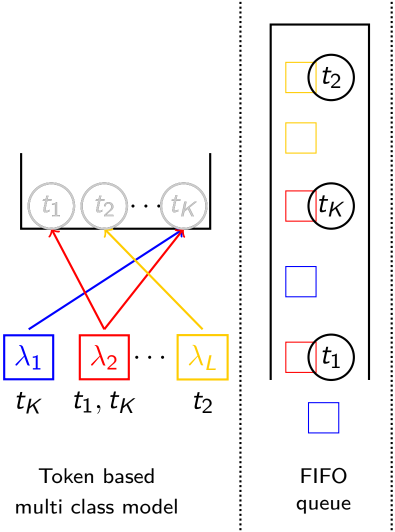
Activating a token



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Applications of OI

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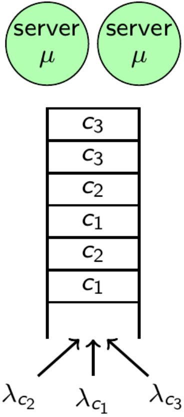

2 identical servers, 3 classes, servers compatible with all classes

Concurrent control: Both servers cannot serve the same class simultaneously

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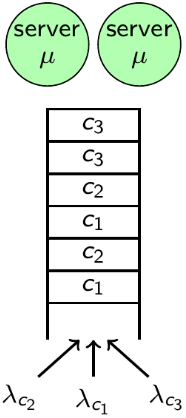
- MSCCC - Multi-server with concurrent customers



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Concurrent control: Both servers cannot serve the same class simultaneously

- Processor-Sharing systems: $\mu(c_1, \ldots, c_n)$ might depend on a scalar function $\phi(n)$