Asymptotically Optimal Load Balancing in Large-scale Heterogeneous Systems with Multiple Dispatchers

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Joint work with...

Ness Shroff, OSU

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Load Balancing...
The Building Block...

Key features:
- Multiple dispatchers
- Heterogeneous servers
Motivating Questions…

1. Question: With multiple dispatchers, does Join-Shortest-Queue still beat others in performance?

▶ No, due to herd behavior, it actually behaves poorly [I. Owen Garrett of NGINX]

▶ Thus, how can we avoid this?

2. Question: Can each dispatcher work independently with simple implementations?

▶ Without communication across dispatchers

3. Question: How much communication between dispatchers and servers?

▶ Minimize the messages between dispatchers and servers

4. Question: Can we say something about performance guarantee?

▶ Stability? or even delay?
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Our Proposed Design Framework: LED

The Local-Estimation-Driven (LED) framework...

1. **Memory:** Each dispatcher has a local memory storing its own estimates of each server’s queue length (often outdated)

2. **Dispatching:** the dispatching decision at each dispatcher is made purely based on local memory

3. **Updating:** the local memory is updated with the true queue length via messages between dispatchers and servers
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Key contributions...

1. Sufficient conditions on dispatching and updating strategies: throughput optimality and delay optimality in heavy traffic
2. Shed light on recently proposed open problem on LB with delayed information [David Lipshutz’19]
Memory: Each dispatcher keeps its own local estimates (often outdated)...

- Dispatcher A ‘believes’ that: server 1 with queue length 5, server 2 with 0, and server 3 with 1
- Dispatcher B ‘believes’ that: server 1 with queue length 4, server 2 with 2, and server 3 with 1
**Dispatching** strategy: Local-Join-Shortest-Queue (L-JSQ)

- each dispatcher independently routes new arrivals to the server with the shortest local estimates
- e.g., Dispatcher A routes to server 2, Dispatcher B routes to server 3
Updating strategy: Push-based update via sampling

- each dispatcher \textit{independently} randomly samples $d$ servers with probability $p$
- update its corresponding local estimates with the true queue lengths
Related Works...

1. LB in multiple dispatchers:
   - JIQ in [Lu et al’ 11]: consider homogeneous servers; JIQ is unstable in general for fixed number of heterogeneous servers [Zhou et al’ 17]
   - Pull-based algorithm in [Stolyar’ 17]: heterogeneous server pools in the large-system regime; assume homogeneous loads across dispatchers
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2. LB with local memory:
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3. Most related to ours is the recent work [Vargaftik et al’ 20]
   - They only consider one particular dispatching strategy, i.e., Local-JSQ.
   - They only investigate stability
Model...

▶ \( M \) dispatchers and \( N \) servers in discrete-time.

▶ **Arrival:** total number of arriving tasks \( A_\Sigma(t) \) with rate \( \lambda_\Sigma \), general distribution \(^1\)
  
  ▶ \( A_\Sigma(t) \) integer-valued \( i.i.d \) across time-slots
  
  ▶ \( A_\Sigma(t) = \sum_{m=1}^{M} A^m(t) \), \( A^m(t) \) arrivals at dispatcher \( m \)
  
  ▶ assume \( \mathbb{P}(A^m(t) > 0) \geq p_0 > 0, \forall (m, t) \in \mathcal{M} \times \mathbb{N} \),

▶ **Service:** average number of tasks can be served at server \( k \) is \( \mu_k \), general distribution.
  
  ▶ \( S_n(t) \) is integer-valued, \( i.i.d \) across time and independent of arrival and queue lengths

▶ **Memory:** \( \tilde{Q}^m(t) = (Q_1^m(t), \ldots, Q_N^m(t)) \)

▶ **System states:** \( Z(t) = (Q(t), \{\tilde{Q}^1(t)\}), \ldots, \tilde{Q}^M(t)) \)

\(^1\)with all moments bounded by absolute constants
In this paper, we consider both throughput optimality and delay optimality in heavy traffic...
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**Definition (Throughput Optimality)**

A LB policy is throughput optimal if the system is positive recurrent under any $\epsilon > 0$ and all the moments of $\|Q^{(\epsilon)}\|$ are finite.

**Note:** this definition is stronger than simple stability.
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A LB policy is throughput optimal if the system is positive recurrent under any \( \epsilon > 0 \) and all the moments of \( \|Q^{(\epsilon)}\| \) are finite

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**Definition (Heavy-traffic Delay Optimality)**

A LB policy is said to be heavy-traffic delay optimal in steady-state if the steady-state queue length vector \( Q^{(\epsilon)} \) satisfies

\[
\lim_{\epsilon \downarrow 0} \epsilon \mathbb{E} \left[ \sum_{n=1}^{N} Q_n^{(\epsilon)} \right] = \lim_{\epsilon \downarrow 0} \epsilon \mathbb{E} [\bar{q}^{\epsilon}],
\]

where \( \mathbb{E} [\bar{q}^{\epsilon}] \) is the mean queue length in resource-pooling system.

**Resource-pooling system:** pool all the service into one super single server
Dispatching Preference...

- Fix a dispatcher $m$, let $\sigma_t(\cdot)$ be a permutation of $(1, 2, \ldots, N)$ that satisfies

\[ \tilde{Q}_{\sigma_t(1)}^m(t) \leq \tilde{Q}_{\sigma_t(2)}^m(t) \leq \cdots \leq \tilde{Q}_{\sigma_t(N)}^m(t). \]

- $P_n^m(t)$: probability of routing to server $n$ at dispatcher $m$ in time-slot $t$ (again, based on local estimates)

- $\Delta_n^m(t)$: preference of the $n$-th shortest local estimate at dispatcher $m$, given by

\[ \Delta_n^m(t) := P_{\sigma_t(n)}^m(t) - \frac{\mu_{\sigma_t(n)}}{\sum \mu_n} \]
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\[ \Delta^m_n(t) := P^m_{\sigma_t(n)}(t) - \frac{\mu_{\sigma_t(n)}}{\sum \mu_n} \]

- $\Delta^m_n(t) > 0$ means that policy has stronger preference of $n$-th shortest local estimates compared to (weighted) random routing

- Note that $\sum_{n=1}^{N} \Delta^m_n(t) = 0$

- **Key:** how to allocate the zero-sum?
\( \delta \)-tilted Sum Condition

\[ \Delta^m_n(t) := P^m_{\sigma_t(n)}(t) - \frac{\mu_{\sigma_t(n)}}{\sum \mu_n} \]

**Definition**

Fix a dispatcher \( m \), for all \( 1 \leq j \leq N - 1 \), \( \sum_{n=1}^{j} \Delta^m_n(t) \geq \delta \) for some constant \( \delta \geq 0 \) at each time-slot \( t \).

**Intuitions:** for any first \( k \ (k < N) \) shortest local estimates, it has at least \( \delta \) total preference
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**Examples:** suppose all \( \mu_n \) are equal and \( \tilde{Q}^m(t) = (5, 0, 1) \)

- \( \delta \)-tilted Sum Condition satisfied with all \( P^m(t) \) s.t. for some \( \delta \geq 0 
- \( P^m_2(t) \geq \delta + 1/3 \), \( P^m_2(t) + P^m_3(t) \geq \delta + 2/3 \), and \( \sum P^m_n(t) = 1 \)

**Implications:**
- this condition also generalizes previous definition in [Zhou et al’ 17,18]
- as a result, it allows us to establish new results (e.g., L-Pod), discussed later
Main Results

We have the following sufficient condition (informal) for throughput optimality...
Define: $I^m_n(t)$ indicates server $n$’s true queue length is updated at dispatcher $m$

**Theorem**
Consider an LED policy if

- dispatching strategy satisfies $\delta$-tilted sum condition for some $\delta \geq 0$
- updating strategy satisfies that $\mathbb{E}[I^m_n(t) \mid Z(t)] > p$ for any $Z(t), m, n$ and some $p > 0$

Then, it is throughput optimal

**Remark:**
- This directly generalizes LSQ policy in [Vargaftik et al’ 20] in terms of stability
Main Results

We have the following sufficient condition (informal) for heavy-traffic delay optimality...

Theorem

Consider an LED policy if

- dispatching strategy satisfies $\delta$-tilted sum condition for some strictly positive constant $\delta$
- updating strategy satisfies that $\mathbb{E} [I_m^m(t) | Z(t)] \geq p > 0$ for any $Z(t), m, n$ independent of previous updates;
- both $\delta$ and $p$ are independent of $\epsilon$

Then, it is heavy-traffic delay optimal
Main Results

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**Theorem**

Consider an LED policy if

- dispatching strategy satisfies $\delta$-tilted sum condition for some strictly positive constant $\delta$
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Then, it is heavy-traffic delay optimal

**Remark:**

- This directly implies a large class of LED policies are heavy-traffic delay optimal, including the specific one LSQ in [Vargaftik et al’ 20]
- This also sheds light on heavy-traffic delay optimality in delayed queue length information, raised in [David Lipshutz’19]
- Moreover, the single dispatcher with accurate information is just a special case of ours
Examples of ‘nice’ dispatching strategies

1. L-JSQ: Local-Join-Shortest-Queue (i.e., the LSQ in [Vargaftik et al’ 20])
   - choose \( i^* \in \arg \min_n \{ \tilde{Q}_n^m \} \)
   - \( \Delta_1^m(t) = 1 - \frac{\mu_{\sigma_{t(1)}}}{\mu_{\Sigma}} > 0 \) and all others are less than 0
   - It can be easily seen that \( \delta \)-tilted sum condition is satisfied
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2. L-JBA: Local-Join-Below-Average
   - Let \( \bar{Q}_m^m(t) = \frac{1}{N} \sum_n \tilde{Q}_n^m(t) \) and \( \mathcal{A} := \{ n : \tilde{Q}_n^m(t) \leq \bar{Q}_m^m(t) \} \)
   - Then, for each \( i \in \mathcal{A} \), \( P_i^m(t) = \frac{\mu_i}{\sum_{n \in \mathcal{A}} \mu_n} \), and for \( i \notin \mathcal{A} \), \( P_i^m(t) = 0. \)
   - it also satisfies the condition, although it needs the information on \( \mu \)
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   - it also satisfies the condition, although it needs the information on $\mu$

3. L-Pod: Local-Power-of-$d$
   - randomly samples $d$ servers, join the one with the shortest local estimates
   - it turns out that even with heterogeneous servers, L-Pod can still satisfy $\delta$-tilted sum as long as the services rates meet a certain condition
Proposition

Suppose the service rate vector $\mu \in \mathbb{R}_+^N$ satisfies

$$\sum_{n=1}^j \frac{\mu[n]}{\mu \Sigma} + \delta \leq 1 - \frac{(N-j)}{\binom{N}{d}} \quad \forall 1 \leq j \leq N - 1,$$

for some constant $\delta \geq 0$, in which $\mu[n]$ is the $n$-th largest service rate. Then, L-Pod satisfies the $\delta$-tilted sum condition.
More on L-Pod

Proposition

Suppose the service rate vector $\mu \in \mathbb{R}^N_+$ satisfies

$$\sum_{n=1}^j \frac{\mu[n]}{\mu \Sigma} + \delta \leq 1 - \frac{{(N-j)}}{{(N \choose d)}} \quad \forall 1 \leq j \leq N - 1, \quad (1)$$

for some constant $\delta \geq 0$, in which $\mu[n]$ is the $n$-th largest service rate. Then, L-Pod satisfies the $\delta$-tilted sum condition.

Remark:

- For the single dispatcher with accurate queue length information (which is a special case of ours), [Hurtado-Lange and Maguluri’ 20]) derived similar conditions
- If $d = 1$, the only possible $\mu$ and $\delta$ are $\mu_n = \mu$ for all $n$ and $\delta = 0$
- If $d = N$, then all $\mu \in \mathbb{R}^N_+$ satisfies (1) with $\delta = \mu_{\text{min}}/\mu \Sigma > 0$
Examples of ‘nice’ updating strategies

We can generally have two categories: push-based and pull-based
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1. Push-based: each dispatcher takes the initiative to sample servers
   - e.g., at the end of each time-slot, w.p. $\tilde{\rho} > 0$ to randomly sample $d$ queues and update the local estimates with the true lengths
   - thus, $\mathbb{E}[I^m_n(t) | Z(t)] \geq p > 0$ is satisfied with $p = \tilde{\rho}d/N$
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We can generally have two categories: push-based and pull-based

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   ▶ e.g., at the end of each time-slot, w.p. $\tilde{p} > 0$ to randomly sample $d$ queues and update the local estimates with the true lengths
   ▶ thus, $\mathbb{E} [\mathcal{I}_n^m(t) \mid Z(t)] \geq p > 0$ is satisfied with $p = \tilde{p}d/N$

2. Pull-based: each server takes the initiative to sample dispatchers
   ▶ e.g., at the end of each time-slot, if server $n$ finishes one or more tasks, it randomly samples one dispatcher
   ▶ if $Q_n = 0$, it reports w.p. 1
   ▶ if $Q_n > 0$, it reports w.p. $\tilde{p} > 0$
   ▶ it has been verified in [Vargaftik et al’ 20]), this satisfies
     $\mathbb{E} [\mathcal{I}_n^m(t) \mid Z(t)] \geq p > 0$ for arbitrarily small $\tilde{p} > 0$

Of course, there are many more...
Recall our motivating questions

1. **Question:** With multiple dispatchers, does Join-Shortest-Queue still beat others in performance?
   - No, due to herd behavior, it actually behaves poorly
   - Thus, how can we avoid this?

**Answer:** LED could be one solution due to its intrinsic randomness
Inaccurate information helps...

100 heterogeneous servers, 10 dispatchers
Randomness further helps...

- 100 homogeneous servers, 10 dispatchers
- update probability is small $\tilde{p} = 0.01$
Recall our motivating questions

2. **Question:** Can each dispatcher work independently with simple implementations?
Recall our motivating questions

2. **Question**: Can each dispatcher work independently with simple implementations?
   - Without communication across dispatchers

**Answer**: For LED, we have
   - each dispatcher totally works independently
   - immediate dispatching, i.e., no waiting for update
   - simple and fast implementations, e.g., min-heap
3. **Question:** How much communication between dispatchers and servers?
Recall our motivating questions

3. Question: How much communication between dispatchers and servers?
   - Minimize the messages between dispatchers and servers

Answer: For LED, we have
   - the sampling and reporting probabilities can be arbitrarily small
   - of course, for practical performance, these parameters can be tuned to trade-off between messages and performance
Recall our motivating questions

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Recall our motivating questions

4. **Question:** Can we say something about performance guarantee?
   - Stability? or even delay?

**Answer:** For LED, we have
   - throughput optimality
   - delay optimality in heavy traffic
Main ideas behind proofs

▶ The main techniques are based on drift-based [Eryilmaz and Srikant’12])

▶ In particular, we utilize the sufficient conditions for throughput and heavy-traffic optimality in [Zhou et al’17], illustrated as follows

Throughput optimality needs positive drift \( \uparrow \), obtained via

\[
\sum_{t=1}^{T} \mathbb{E} \left[ \langle Q, A - S \rangle \mid Q \right] \approx -\epsilon \|Q\|.
\]

Heavy-traffic optimality needs positive drift \( \downarrow \), obtained via

\[
\sum_{t=1}^{T} \mathbb{E} \left[ \langle Q_\perp, A - S \rangle \mid Q \right] \approx -\delta \|Q_\perp\|.
\]
Main ideas behind proofs

Three additional challenges arise in our settings...

1. A more general dispatching condition (i.e., $\delta$-tilted sum condition)
   ▶ it exists even when the queue lengths are accurate
   ▶ we draw inspirations from [Hurtado-Lange and Maguluri’ 20] to have a nice bound on the inner product between $Q$ and $A$

2. Outdated queue lengths information
   ▶ our strategy is to do a decomposition
   ▶ first, establish necessary drifts via dispatching strategy, assuming the queue lengths are accurate
   ▶ second, bounding the error via update condition

3. System state includes local estimates
   ▶ hence, for throughput optimality, they should also be bounded
Conclusion...

The LED combined with sufficient conditions give affirmative answers to all key questions...

1. **Question**: With multiple dispatchers, does Join-Shortest-Queue still beats others in performance?
   **Answer**: LED could be one solution due to its intrinsic randomness

2. **Question**: Can each dispatcher work independently with simple implementations?
   **Answer**: LED achieve independence, easy implementations

3. **Question**: How much communication between dispatchers and servers?
   **Answer**: LED, has the flexibility to tune the probability $\tilde{p}$

4. **Question**: Can we say something about performance guarantee?
   **Answer**: LED, can be throughput optimal and delay optimal in heavy traffic
There are several interesting directions for LED...

1. Beyond the traditional heavy-traffic regime?
   - As pointed out by [Zhou et al’ 18]), heavy-traffic delay optimal is a coarse metric in certain sense
   - How about waiting probability in large-system regimes?

2. How about continuous-time systems?

3. How about LED on graphs?
   - each node can serve a job or dispatches to neighbors
   - each node keeps local estimates of its neighbors
   - purely based on local memory to dispatch
   - infrequent update via communications between nodes
Thank you!

Q & A
Throughput optimality...

1. We consider the Lyapunov function
   \[ W(Z(t)) = \|Q(t)\|^2 + \sum_{m=1}^{M} \|Q(t) - \tilde{Q}^m(t)\|_1 \]

2. Let \( X^m_n(t) \triangleq |Q_n(t) - \tilde{Q}^m_n(t)| \), the drift is

   \[ D(Z(t_0)) = D_Q(t_0) + \sum_{m=1}^{M} \sum_{n=1}^{N} D_{X^m_n}(t_0) \quad (2) \]

   where

   \[ D_Q(t_0) \triangleq \mathbb{E} \left[ \|Q(t_0 + T)\|^2 - \|Q(t_0)\|^2 \mid Z(t_0) \right] \]

   \[ D_{X^m_n}(t_0) \triangleq \mathbb{E} \left[ X^m_n(t_0 + T) - X^m_n(t_0) \mid Z(t_0) \right] \]

3. \( D_{X^m_n}(t_0) \leq -pX^m_n(t) + 2T \mu \Sigma \)
4. Turn to $D_Q(t_0) \triangleq \mathbb{E} \left[ \| Q(t_0 + T) \|^2 - \| Q(t_0) \|^2 \mid Z(t_0) \right] \approx \sum \mathbb{E} \left[ \langle Q, A - S \rangle \mid Z(t_0) \right] + K$

5. We can decompose the first term into $(\beta^m_n(t) := P^m_n(t) - \mu_n/\mu_{\Sigma})$

$$\text{RHS} \approx \sum_{t=t_0}^{t_0+T-1} \mathbb{E} \left[ \sum_{n=1}^{N} \sum_{m=1}^{M} (Q_n(t) - \tilde{Q}_n^m(t)) \beta^m_n(t) \lambda_m \mid Z \right]$$

$$\text{RHS} \approx \sum_{t=t_0}^{t_0+T-1} \mathbb{E} \left[ \sum_{n=1}^{N} \sum_{m=1}^{M} \tilde{Q}_n^m(t) \beta^m_n(t) \lambda_m \mid Z \right] - \frac{\epsilon \mu_{min}}{\mu_{\Sigma}} \| Q(t_0) \|_1 .$$

6. For $T_1$, by update condition, we have a constant bound on it
Throughput optimality (Cont’d)

7. Turn to $T_2 = \sum_{t=t_0}^{t_0+T-1} \mathbb{E} \left[ \sum_{n=1}^{N} \sum_{m=1}^{M} \tilde{Q}_n^m(t) \beta_n^m(t) \lambda_m | Z \right]$

8. It is equal to $\sum_{t=t_0}^{t_0+T-1} \mathbb{E} \left[ \sum_{n=1}^{N} \sum_{m=1}^{M} \tilde{Q}_{\sigma_t(n)}^m(t) \Delta_n^m(t) \lambda_m | Z \right]$

9. The green term can be written as

$$\sum_{m=1}^{M} \left( \tilde{Q}_{\sigma_t(1)}^m(t) \sum_{n=1}^{N} \Delta^m_n(t) \right)$$  \hspace{1cm} (3)

$$+ \sum_{m=1}^{M} \left( \sum_{k=2}^{N} \left( \sum_{n=k}^{N} \Delta^m_n(t) \right) \left( \tilde{Q}_{\sigma_t(k)}^m(t) - \tilde{Q}_{\sigma_t(k-1)}^m(t) \right) \right)$$  \hspace{1cm} (4)

10. (3) is zero as $\sum_{n=1}^{N} \Delta^m_n(t) = 0$

11. (4) less than zero since $\sum_{n=k}^{N} \Delta^m_n(t) \leq -\delta$ by $\delta$-tilted sum condition
Heavy-traffic delay optimality...

1. We wish to establish
\[ \sum_{t=t_0}^{t_0+T-1} \mathbb{E} \left[ \langle Q_\perp, A - S \rangle \mid Z \right] \approx -\delta' \| Q_\perp \|, \delta' \text{ independent of } \epsilon \]

2. The key term \( \sum_{t=t_0}^{t_0+T-1} \mathbb{E} \left[ \langle Q_\perp, A \rangle \mid Z \right] \) can be written as
\[
\sum_{t=t_0}^{t_0+T-1} \mathbb{E} \left[ \sum_{n=1}^{N} Q_{\perp,n}(t) \sum_{m=1}^{M} \beta_{m}^{n}(t) \lambda_{m} \mid Z \right]
\]
\[
= \sum_{t=t_0}^{t_0+T-1} \mathbb{E} \left[ \sum_{n=1}^{N} \sum_{m=1}^{M} \left( \tilde{Q}_{n}^{m}(t) - \bar{Q}_{m}(t) \right) \beta_{n}^{m}(t) \lambda_{m} \mid Z \right] \quad (5)
\]
\[
+ \sum_{t=t_0}^{t_0+T-1} \mathbb{E} \left[ \sum_{n=1}^{N} \sum_{m=1}^{M} \left( Q_{n}(t) - \tilde{Q}_{n}^{m}(t) \right) \beta_{n}^{m}(t) \lambda_{m} \mid Z \right] \quad (6)
\]
\[
+ \sum_{t=t_0}^{t_0+T-1} \mathbb{E} \left[ \sum_{n=1}^{N} \sum_{m=1}^{M} \left( \tilde{Q}_{n}^{m}(t) - Q_{\text{avg}}(t) \right) \beta_{n}^{m}(t) \lambda_{m} \mid Z \right] \quad (7)
\]

where \( \tilde{Q}_{n}^{m}(t) := \frac{1}{N} \sum_{n} \tilde{Q}_{n}^{m}(t) \) and \( Q_{\text{avg}} := \frac{1}{N} \sum_{n} Q_{n}(t) \)

3. By updating condition, (6) and (7) both can be upper bounded (properly chosen \( T \))
Heavy-traffic delay optimality (Cont’d)

4. Turn to the green term, it can be written as

$$
(5) = \sum_{t=t_0}^{t_0+T-1} \mathbb{E} \left[ \sum_{m=1}^{M} \sum_{n=1}^{N} \tilde{Q}^m_{\sigma_t(n)}(t) \Delta^m_n(t) \lambda_m \mid Z \right]
$$

5. Follow the same decompositions as in (3) and (4), we have

$$
(5) \leq -\delta \sum_{t=t_0}^{t_0+T-1} \mathbb{E} \left[ \sum_{m=1}^{M} \sum_{n=1}^{N} \tilde{Q}^m_{max}(t) - \tilde{Q}^m_{min}(t) \lambda_m \mid Z \right]
$$

6. By a careful sample-path analysis, we have for some constant $K$

$$
(5) \leq -\delta f(p) \lambda_{min} (Q^m_{max}(t_0) - Q_{min}(t_0)) + K \\
\leq -\delta' \|Q_{\perp}(t_0)\| + K
$$